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Étale Endomorphisms of Smooth Affine Surfaces

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1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero. Let $k[x, y]$ be a polynomial ring in two variables x, y over k, let $A = k[x, y, f^{-1}]$ for an irreducible element f of $k[x, y]$, and let $X = \operatorname{Spec} A$. The purpose of the present article is to determine a pair (X, f) with an étale endomorphism $\alpha\colon X \to X$ which is not an automorphism. Choose such a pair (X, f) with an étale endomorphism α of X as an algebraic variety.

Since α is not an automorphism, $\bar{\kappa}(X) < 2$ by the following result of Iitaka [10]:

LEMMA 1.1. *Let X be a nonsingular algebraic variety with logarithmic Kodaira dimension equal to* $\dim X$. *Let $\alpha\colon X \to X$ be a quasi-finite endomorphism. Then α is an automorphism.*

A main part of the present article is devoted to classifying all the pairs (X, f) as above up to isomorphisms when the logarithmic Kodaira dimension $\bar{\kappa}(X) \leq 1$ and to determining all étale endomorphisms associated to each possible pair (X, f) in the classification list. In fact, there is one case (case (3) of Theorem 3.7) where we have to assume an extra condition. Among others, we mention the following two results which will be proved later and which concern the existence of étale finite endomorphisms:

THEOREM 1.2. *Assume that $\bar{\kappa}(X) = -\infty$. After a suitable change of coordinates of $k[x, y]$, we have $f = x$. Then an étale endomorphism α of X*

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has the following associated algebra homomorphism $\alpha^*\colon k[x, y, 1/x] \to k[x, y, 1/x]$

$$\alpha^*(x) = cx^n, \qquad \alpha^*(y) = \frac{x^m}{x^r y + g(x)},$$

where $c \in k$, $c \neq 0$, $r \in \mathbf{Z}$, $r \geq 0$, *and* $g(x) \in k[x]$. *The endomorphism* α *is then a finite étale endomorphism.*

THEOREM 1.3. *Suppose* $\bar{\kappa}(X) \geq 0$. *Then* X *has no étale finite endomorphisms unless* X *is isomorphic to the surface* $\mathbf{A}^2 - F_0$, *where* F_0 *is a curve on* $\mathbf{A}^2$ *defined by* $y^m - x^n = 0$ *for positive integers* m, n *with* $\gcd(m, n) = 1$. *For this surface* X, *all étale endomorphisms of* X *are finite, and there exists an étale endomorphism of* X *whose degree exceeds an arbitrarily given integer.*

We fix the notation and give some of the results to be used later. For $\lambda \in k$, F_λ denotes an affine curve defined by $f = \lambda$. Then a family of curves $\Lambda = \{F_\lambda;\ f = \lambda\ (\lambda \in k)\}$ is called a linear pencil on $\mathbf{A}_k^2$ defined by f. Let ϕ $\mathbf{A}^2 \to \mathbf{A}^1$ be the associated morphism. Then ϕ_Λ restricted on X induces a morphism $\varphi\colon X \to {\mathbf{A}^1}_*$.

We recall the following well-known result [10, 11].

LEMMA 1.4. *Let* $\sigma\colon X \to Y$ *be a dominant morphism of algebraic varieties* X, Y. *Assume that a general fiber* $X_y = \sigma^{-1}(y)$ *of* σ *over a general point* $y \in Y$ *is an irreducible curve. Then we have the following inequalities for the logarithmic Kodaira dimensions*:

$$\bar{\kappa}(X) \geq \bar{\kappa}(Y) + \bar{\kappa}(X_y) \tag{1}$$

$$\dim Y + \bar{\kappa}(X_y) \geq \bar{\kappa}(X). \tag{2}$$

Applying Lemma 1.4 to the morphism $\varphi\colon X \to {\mathbf{A}^1}_*$, we obtain the inequalities

$$1 + \bar{\kappa}(F) \geq \bar{\kappa}(X) \geq \bar{\kappa}({\mathbf{A}^1}_*) + \bar{\kappa}(F).$$

2. CASE $\bar{\kappa}(X) = -\infty$

First of all, we remark that the hypothesis that f is irreducible is not necessary in the present case $\bar{\kappa}(X) = -\infty$. In fact, we have the following result [14].

LEMMA 2.1. *Suppose that* $\bar{\kappa}(X) = -\infty$ *and that one of the following conditions is satisfied*:

(1) X *is irrational but not elliptic ruled*;

(2) $\Gamma(X, \mathcal{O}_X)^* \neq k^*$ *and* $\operatorname{rank}(\Gamma(X, \mathcal{O}_X)^*/k^*) \geq 2$ *if* X *is rational.*

Then an étale endomorphism $f\colon X \to X$ *is an automorphism.*

Since $X = \mathbf{A}^2_k - F_0$, X is a rational surface and $\Gamma(X, \mathscr{O}_X)^*/k^*$ is generated by prime factors of f. By the above lemma, rank $(\Gamma(X, \mathscr{O}_X)^*/k^*) = 1$ provided α is not an automorphism. So, f is irreducible. Furthermore, since $\mathbf{P}^1 - \{0, \infty\} = \mathbf{A}^1{}_*$ and $\overline{\kappa}(\mathbf{A}^1{}_*) = 0$, we have $\overline{\kappa}(F) = -\infty$ by the formula (1), where F is a general fiber of $\varphi\colon X \to \mathbf{A}^1{}_*$. Furthermore, the second theorem of Bertini implies that F is a smooth curve. Hence $F \cong \mathbf{A}^1$. Here, we shall use the embedded line theorem of Abhyanker-Moh [1].

LEMMA 2.2. *Let $C \subset \mathbf{A}^2$ be an irreducible curve isomorphic to $\mathbf{A}^1$. Then there exist coordinates u, v on $\mathbf{A}^2$ such that $k[x, y] = k[u, v]$ and C is defined by $u = 0$.*

By the above lemma, we may assume that a general fiber F of φ, i.e., F_λ for $\lambda \in k$, is defined by $u = 0$. Then the curve F_0 defined by $f = 0$ is defined by $u + \lambda = 0$. So, $F_0 \cong \mathbf{A}^1$. We may assume that $f = x$. Hence, after a change of variables, the coordinate ring A of X is isomorphic to $k[x, y, 1/x]$. Then $X \cong \mathbf{A}^1{}_* \times \mathbf{A}^1$.

We shall show that α^* is written as stated in Theorem 1.2, where α^* is the algebra homomorphism of the coordinate ring corresponding to a given étale endomorphism α of X. Note that the multiplicative group A^* of invertible elements of A is generated by k^* and the element x. Since α^* induces a multiplicative group homomorphism from A^* to A^*, we may write

$$\alpha^*(x) = cx^n, \qquad \alpha^*(y) = \frac{\varphi(x, y)}{x^m},$$

where $\varphi(x, y) \in k[x, y]$, $c \in k^*$, $n, m \in \mathbf{Z}$, $n \neq 0$, $m \geq 0$. Then we obtain the following matrix relation for the differentials:

$$\begin{pmatrix} d\alpha^*(x) \\ d\alpha^*(y) \end{pmatrix} = \begin{pmatrix} ncx^{n-1} & 0 \\ \dfrac{\varphi_x x^m - m\varphi x^{m-1}}{x^{2m}} & \dfrac{\varphi_y}{x^m} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Since α^* is unramified, the determinant of this matrix must be an invertible element of A. Hence we have

$$ncx^{n-1}\frac{\varphi_y}{x^m} = c'x^{\ell} \qquad \text{for some } c' \in k^* \text{ and } \ell \in \mathbf{Z}.$$

Namely, we have

$$\varphi_y = \frac{c'}{nc} x^{l+m-n+1} \qquad \text{with } l+m-n+1 \geq 0.$$

This implies that

$$\varphi \sim x^r y + g(x) \qquad \text{with } r = l+m-n+1 \geq 0 \text{ and } g(x) \in k[x],$$

where $\sim$ means that $\varphi(x, y)$ is determined by the right side up to an element of k^*. Furthermore, we shall show that $k[x, y, 1/x]$ is integral over $k[\alpha^*(x), \alpha^*(y), 1/\alpha^*(x)]$. From the above arguments, we have

$$k\left[\alpha^*(x), \alpha^*(y), \frac{1}{\alpha^*(x)}\right] = k\left[x^n, \frac{x^r y + g(x)}{x^m}, \frac{1}{x^n}\right].$$

Note that

$$k\left[x, \frac{x^r y + g(x)}{x^m}, \frac{1}{x}\right] = k\left[x, y, \frac{1}{x}\right].$$

Hence $k[x, y, x^{-1}]$ is integral over $k[\alpha^*(x), \alpha^*(y), \alpha^*(x)^{-1}]$. So, α^* is a finite étale endomorphism. This completes a proof of Theorem 1.2.

3. CASE $\bar{\kappa}(X) \geq 0$

Since $X = \mathbf{A}^2 - F_0$, X is a smooth affine rational surface. Furthermore, since $A = k[x, y, 1/f]$ is factorial, it follows that $\operatorname{Pic}(X) = 0$.

We consider first the case $\bar{\kappa}(X) = 0$. Applying the addition formula (1) of Lemma 1.4 to the morphism φ: $X \to \mathbf{A}^1{}_*$, we have

$$0 = \bar{\kappa}(X) \geq \bar{\kappa}(F) + \bar{\kappa}(\mathbf{A}^1{}_*),$$

where F is a general fiber of φ. If $\bar{\kappa}(F) = -\infty$ then F is isomorphic to $\mathbf{A}^1$ and φ becomes an $\mathbf{A}^1$-fibration, which implies that $\bar{\kappa}(X) = -\infty$, a contradiction. So, $\bar{\kappa}(F) = 0$ because $\bar{\kappa}(\mathbf{A}^1{}_*) = 0$. Hence φ is an $\mathbf{A}^1{}_*$-fibration. In accordance with the later use of notation, we denote it by ρ: $X \to B$, where $B \cong \mathbf{A}^1{}_*$.

Next consider the case $\bar{\kappa}(X) = 1$. We need the following lemma of Kawamata [12]; see also Gurjar and Miyanishi [8, Lemma 10]:

LEMMA 3.1. *Let X be an affine smooth rational surface with $\bar{\kappa}(X) = 1$. Then there exists a morphism ρ: $X \to B$ onto a nonsingular rational curve B which defines a twisted or untwisted $\mathbf{A}^1{}_*$-fibration.*

So, we have an $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ in both cases $\bar{\kappa}(X) = 0$ and $\bar{\kappa}(X) = 1$. In the case where ρ is a twisted $\mathbf{A}^1{}_*$-fibration, $\mathrm{Pic}(X)$ contains a nonzero 2-torsion element [7, Proof of Lemma 3.2]. Since $\mathrm{Pic}(X) = (0)$ in the present case, the $\mathbf{A}^1{}_*$-fibration ρ must be untwisted.

We shall show in the case $\bar{\kappa}(X) = 1$ that

$$B \cong \mathbf{P}^1, \mathbf{A}^1 \text{ or } \mathbf{A}^1{}_*.$$

Since B is rational, B is isomorphic to $\mathbf{P}^1$ minus n points. Since f is irreducible, rank $\Gamma(X, \mathscr{O}_X)^*/k^* = 1$. If $n \geq 3$, then rank $\Gamma(B, \mathscr{O}_B)^*/k^* \geq 2$. Since $\Gamma(B, \mathscr{O}_B)^*/k^*$ is a subgroup of $\Gamma(X, \mathscr{O}_X)^4/k^*$, this is a contradiction. So, $n \leq 2$. This implies the above assertion. Before going further, we present the following result:

LEMMA 3.2. *Let $\rho\colon X \to B$ be the $\mathbf{A}^1{}_*$-fibration as above. Let $\alpha\colon X \to X$ be a dominant endomorphism. In each of the following cases, there exists then an endomorphism $\beta\colon B \to B$ such that $\rho \circ \alpha = \beta \circ \rho$:*

(1) *The morphism ρ coincides with the morphism φ defined by the pencil $\{f = \lambda;\ \lambda \in k^*\}$.*

(2) $\bar{\kappa}(X) = 1$ *and α is finite and étale.*

Proof. (1) Let $\alpha^*\colon A \to A$ be the associated k-homomorphism. Since $\alpha^*(f)$ is invertible in A, we may write $\alpha^*(f) = cf^n$ with $c \in k^*$ and $n \neq 0$. Define an endomorphism $\beta\colon \operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[t, t^{-1}]$ by $\beta^*(t) = ct^n$. Then $\rho \circ \alpha = \beta \circ \rho$.

(2) Note that in the case $\bar{\kappa}(X) = 1$ the $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ is uniquely determined by the surface X by virtue of Lemma 3.3 below. Let F be a general fiber of ρ. Since $F \cong \mathbf{A}^1{}_*$ and since the pull-back of $\mathbf{A}^1{}_*$ by an étale finite morphism is a disjoint union of the curves isomorphic to $\mathbf{A}^1{}_*$, it follows from the uniqueness of the $\mathbf{A}^1{}_*$-fibration on X that $\alpha^{-1}(F)$ consists of finitely many fibers of ρ for the upper X.[1] Let $\tilde{\beta}\colon \tilde{B} \to B$ be the normalization of B in the function field of the upper X. Then there exists a morphism $\tilde{\rho}\colon X \to \tilde{B}$ such that $\rho \circ \alpha = \tilde{\beta} \circ \tilde{\rho}$ and the general fibers of $\tilde{\rho}$ are isomorphic to $\mathbf{A}^1{}_*$ by the above observation. By using the uniqueness of the $\mathbf{A}^1{}_*$-fibration on X we know that $\tilde{B} = B$ and $\tilde{\rho} = \rho$. Hence a general fiber F_u on the upper X maps to a general fiber of ρ on the lower X. Q.E.D.

We made use of the following result in the proof of the previous lemma.

LEMMA 3.3. *Let X be a smooth algebraic surface with* $\bar{\kappa}(X) = 1$, *and let $\rho\colon X \to B$ be an $\mathbf{A}^1{}_*$-fibration. Let (V, D) be a pair of a smooth projective*

[1] For an endomorphism $\alpha\colon X \to X$ we call the source X (resp. the target X) the upper (resp. the lower) X.

surface and an effective reduced divisor with simple normal crossings such that $V - D = X$ *and the* $\mathbf{A}^1{}_*$*-fibration extends to a* $\mathbf{P}^1$*-fibration* $p\colon V \to C$, *where* C *is a smooth projective curve containing* B *as an open set. Then* $\dim|n(D + K_V)| > 0$ *for some* $n > 0$ *and the movable part of* $|n(D + K_V)|$ *is composed of the pencil associated with the fibration* $p\colon V \to C$.

Proof. By the hypothesis $\bar{\kappa}(X) = 1$ there exist a pair (V, D) as above and an integer $n > 0$ such that $\dim|n(D + K_V)| > 0$ and the movable part M of the linear system $|n(D + K_V)|$ is composed of a pencil. It suffices to show that $(\overline{F} \cdot G) = 0$ for a general fiber $\overline{F}$ of p and a general member G of M. Note that the fiber $\overline{F}$ is the closure of a general fiber F of ρ in V and that $\overline{F}$ consists of F and two smooth points on D. Hence $(\overline{F} \cdot D) = 2$. Since $(\overline{F}^2) = 0$, we have $(\overline{F} \cdot K_V) = -2$. Hence $(\overline{F} \cdot n(D + K_V)) = 0$. Write $|n(D + K_V)| = M + H$, where H is the fixed part. Then $(\overline{F} \cdot M) \geq 0$ and $(\overline{F} \cdot H) \geq 0$. Hence we have $(\overline{F} \cdot G) = 0$. Q.E.D.

Now we shall determine the standard forms of f. We have two cases which we consider below separately.

(I) Case ρ extends to an $\mathbf{A}^1{}_*$-fibration on $\mathbf{A}^2$,

(II) Case ρ does not extend to an $\mathbf{A}^1{}_*$-fibration on $\mathbf{A}^2$.

In the case (I), let $\tilde{\rho}\colon \mathbf{A}^2 \to \tilde{B}$ be the extension of ρ. Then F_0 cannot meet the general fibers of $\tilde{\rho}$. Indeed, if it meets a general fiber, some more points are deleted from $\mathbf{A}^1{}_*$, which is not the case. So, F_0 is isomorphic to either $\mathbf{A}^1{}_*$ or an irreducible component of a singular fiber of $\tilde{\rho}$. We call an element $f \in k[x, y]$ *a generically rational polynomial with n places at infinity* if a general fiber of the morphism $\phi_\Lambda\colon \mathbf{A}^2 \to \mathbf{A}^1$ associated with a pencil $\Lambda = \{f = \lambda;\ \lambda \in k\}$ is a rational curve with n places at infinity. Assume that $F_0 \cong \mathbf{A}^1{}_*$. Then f is a generically rational polynomial with two places at infinity. We recall the following lemma of Saito [13, Theorem 2.3; 16].

LEMMA 3.4. *Let f be a generically rational polynomial in* $k[x, y]$ *with two places at infinity. Then, after a suitable change of coordinates, f is reduced to either one of the following two forms*:

(1) $f \sim x^a y^b + 1$, *where* $a, b > 0$ *and* $\gcd(a, b) = 1$.

(2) $f \sim x^a(x^l y + p(x))^b + 1$, *where* $a, b, l > 0$, $\gcd(a, b) = 1$ *and* $p(x) \in k[x]$ *with* $\deg p(x) < l$ *and* $p(0) \neq 0$.

These two cases in the above lemma give two cases to consider in the case (I).

Next suppose that F_0 is isomorphic to an irreducible component of a singular fiber of $\tilde{\rho}$. Given an $\mathbf{A}^1{}_*$ fibration, we can classify all possible types of singular fibers by the following lemma of Miyanishi [14].

LEMMA 3.5. *Let ρ: $X \to B$ be an $\mathbf{A}^1{}_*$-fibration on an affine smooth surface X over a smooth curve B, and let S be a singular fiber of ρ. Then S is written as a divisor in the form $S = \Gamma + \Delta$, where*

(1) $\Gamma = 0$ *or* $\Gamma = a\Gamma_1$ *with* $a \geq 1$ *and* $\Gamma_1 \cong \mathbf{A}^1{}_*$ *or* $\Gamma = a_1\Gamma_1 + a_2\Gamma_2$, *where* $a_1 \geq 1$, $a_2 \geq 1$, $\Gamma_1 \cong \Gamma_2 \cong \mathbf{A}^1$ *and* Γ_1 *and* Γ_2 *meet each other transversally in one point*;

(2) $\Delta \geq 0$ *and* $\operatorname{Supp}\Delta$ *is a disjoint union of connected components isomorphic to* $\mathbf{A}^1$ *provided* $\Delta > 0$.

From the above result, F_0 is isomorphic to $\mathbf{A}^1{}_*$ or $\mathbf{A}^1$. Note that a general member of $\bar{\rho}$: $\mathbf{A}^2 \to \overline{B}$ is defined by $g = 0$, where g is a generically rational polynomial with two places at infinity. Hence we may assume that g has one of the forms in Lemma 3.4. The singular fiber of $\bar{\rho}$ containing F_0 as an irreducible component is defined by $x^a y^b = 0$ or $x^a(x^\ell y + p(x))^b = 0$. If $F_0 \cong \mathbf{A}^1{}_*$, then f is an irreducible factor $x^\ell y + p(x) = 0$, which is a generically rational polynomial with two places at infinity. If $F_0 \cong \mathbf{A}^1$, we have $A = k[x, y, 1/x]$ by the theorem of Abhyankar and Moh and $X \cong \mathbf{A}^1 \times \mathbf{A}^1{}_*$. Hence we have $\bar{\kappa}(X) = -\infty$, which is a contradiction. So, the case $F_0 \cong \mathbf{A}^1$ is excluded.

Next we consider the case (II) where $\tilde{\rho}$ does not extend to an $\mathbf{A}^1{}_*$-fibration on $\mathbf{A}^2$. We have the following two cases to consider:

(1) The case where ρ can be extended to an $\mathbf{A}^1$-vibration $\tilde{\rho}$: $\mathbf{A}^2 \to B$. Then F_0 is necessarily a cross section of $\tilde{\rho}$. Indeed, $\tilde{\rho}|_{F_0}$: $F_0 \to B$ is injective and F_0 is smooth. Since the general fiber of $\tilde{\rho}$ is isomorphic to $\mathbf{A}^1$, we may assume that it is defined by $x = \lambda$. The curve $x = \lambda$ meets the cross-section $f(x, y) = 0$ at one point. So, $f(\lambda, y) = 0$ has only one solution. Write

$$f(x, y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x),$$

where $a_0(x), \ldots, a_n(x) \in k[x]$ and $a_0(x) \neq 0$. Then $a_0(\lambda) \neq 0$ for a general element $\lambda \in k$. Hence we have $n = 1$. So, $f(x, y) = a_0(x)y + a_1(x)$, w here $\gcd(a_0(x)), a_1(x)) = 1$ because $f(x, y)$ is irreducible. Write $a_1(x) = q(x)a_0(x) + b_1(x)$ with $\deg b_1(x) < \deg a_0(x)$. By replacing y by $y + q(x)$, we may assume that $\deg a_1(x) < \deg a_0(x)$. Suppose $a_0(x)$ has a single linear factor. Then we may assume that $a_0(x) = x^m$ after a change of variable. Write

$$a_1(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-r}x^r + c,$$

where $b_0, b_{n-r}, c \in k^*$ and $m > n \geq r > 0$ if $\deg a_1(x) > 0$. Then f is written as

$$f = x^r(x^{m-r}y + p(x)) + c,$$

where $\deg p(x) < m - r$ and $p(0) \neq 0$ provided $\deg a_1(x) > 0$. Hence f is reduced to either one of the cases (1) and (2) of Lemma 3.4. So, we may assume furthermore that $a_0(x)$ has two or more distinct linear factors.

(2) The case where the closures of the general fibers of ρ in $\mathbf{A}^2$ have one common point P which lies on the curve F_0. Let $\bar{S}$ be the closure of a general fiber S of ρ. Then $\bar{S} = S \cup \{P\}$, where the point P is a one-place point of $\bar{S}$. Therefore $\bar{S}$ is a topologically contractible curve. Then the following lemma of Lin and Zaidenberg (cf. [9]) is available:

LEMMA 3.6. *Let $C \subset \mathbf{A}^2$ be an irreducible algebraic curve which is topologically contractible.*[2] *Then there exist affine coordinates x, y on $\mathbf{A}^2$ such that in terms of these coordinates C is defined by an equation $x^m = y^n$, where* $\gcd(m, n) = 1$.

So, we may assume that $\bar{S}$ is defined by $x^m - \lambda y^n = 0$, where $\lambda \in k$, and the point P is the point of origin. If λ moves in $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$, the curves $\{x^m - \lambda y^n = 0;\ \lambda \in \mathbf{P}^1\}$ is a linear pencil on $\mathbf{A}^2$ parametrized by $\mathbf{P}^1$. Since the curve F_0 is irreducible and reduced, we may assume that $f = x^m - y^n$.

Summarizing the above arguments, we have

THEOREM 3.7. *Suppose that $\bar{\kappa}(X) \geq 0$. Then, after a suitable change of coordinates x, y, the polynomial $f(x, y)$ is reduced to one of the following forms*:

(I) *The case where the given $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ extends to an $\mathbf{A}^1{}_*$-fibration $\tilde{\rho}\colon \mathbf{A}^2 \to \tilde{B}$.*

(1) $f \sim x^a y^b + 1$, *where $a, b > 0$ and* $\gcd(a, b) = 1$. *In this case, $\tilde{B} \cong \mathbf{A}^1$ and $B \cong \mathbf{A}^1{}_*$.*

(2) $f \sim x^a(x^l y + p(x))^b + 1$, *where $a, b, l > 0$, $\gcd(\alpha, \beta) = 1$, and $p(x) \in k[x]$ with* $\deg p(x) < l$ *and*$p(0) \neq 0$. *In this case, $\tilde{B} \cong \mathbf{A}^1$ and $B \cong \mathbf{A}^1{}_*$.*

(II) *The case where the given $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ is no extended to an $\mathbf{A}^1{}_*$-fibration on $\mathbf{A}^2$.*

(3) $f \sim a_0(x)y + a_1(x)$, $\gcd(a_0(x), a_1(x)) = 1$, *where* $\deg a_1(x) < \deg a_0(x)$ *and $a_0(x)$ has two or more distinct linear factors. In this case, the $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ extends to an $\mathbf{A}^1$-fibration $\tilde{\rho}\colon \mathbf{A}^2 \to \tilde{B}$, where $\tilde{B} = B \cong \mathbf{A}^1$.*

(4) $f \sim x^m - y^n$, $\gcd(m, n) = 1$. *In this case, the closures of the fibers of the $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ form a linear pencil $\{x^m - \lambda y^n = 0\}$*

[2] In algebraic terms, we may say that C is a rational curve with a unique place at infinity and a unique one-place singular point P.

parametrized by $\lambda \in \mathbf{P}^1 = k \cup \{\infty\}$, *which has the point origin as a base point. Furthermore,* $\tilde{B} \cong \mathbf{P}^1$ *and* $B \cong \mathbf{A}^1$.

In the rest of this section, we determine the Kodaira dimension of X in each of the above four cases. We recall necessary definitions and results. Let V be a nonsingular projective surface and let D be a reduced effective divisor with simple normal crossings. We set $X := V - D$. Let P be a point on D lying on one and only one irreducible component of D. Let σ: $V' \to V$ be the blowing-up of the point P and let E be the resulting (-1) curve on V'. Let D' be the proper transform of D on V' and let $X' = V' - D'$. We say that X' is obtained from X by a *half-point attachment*. We also say that X is obtained from X' by a *half-point detachment*. See [6] for the definition.

LEMMA 3.8. *In the case* (1) *of Theorem* 3.7, $\bar{\kappa}(X) = 1$ *provided* $a > 1$ *and* $b > 1$, *while* $\bar{\kappa}(X) = 0$ *provided* $a = 1$ *or* $b = 1$.

Proof. Let

$$F(X,Y,Z) = X^a Y^b + Z^{a+b}$$

be the homogenization of the polynomial $f = x^a y^b + 1$ and let C be the curve on $\mathbf{P}^2$ defined by $F(X,Y,Z) = 0$. Then $X = \mathbf{P}^2 - (C + \ell_\infty)$, where ℓ_∞ is the line at infinity. The pencil $\Lambda = \{f_\lambda;\ f = \lambda\ (\lambda \in k)\}$ is cut out on $\mathbf{A}^2$ by the pencil L on $\mathbf{P}^2$, which is spanned by C and $(a+b)\ell_\infty$. The pencil L has the base points centered at the points $(1,0,0)$ and $(0,1,0)$. We shall eliminate the base points. The blowing-ups centered at these base points do not change the affine part X. Let V be the projective surface obtained by these blowing-ups and let D be the boundary of X, i.e., $D = V - X$. Then the proper transform $\tilde{L}$ of L defines a $\mathbf{P}^1$-fibration p: $V \to \mathbf{P}^1$ on V and D contains two cross sections M_0, M_1 and two complete fibers G_0 and G_∞ of the $\mathbf{P}^1$-fibration which come from the curve C and the line at infinity ℓ_∞. Furthermore, the curve $f = 1$ on X gives rise to a third (singular) fiber G_1 of the fibration p. Note that the contraction of a (-1) curve in D, which does not break the normal crossing property, does not change the Kodaira dimension of X (cf. [15]). Similarly, the blowing-up of a point on D does not alter the Kodaira dimension. Hence, by applying the blowing-ups and blowing-downs, especially the elementary transformation for a $\mathbf{P}^1$-fibration, we may assume that the fibers G_0 and G_∞ are smooth fibers, that $(M_0^2) = 0$ and $(M_1^2) \le -1$, and that the singular fiber G_1 has a linear chain as its dual graph and contains no (-1) curves except possibly for the components C_1, C_2 defined by $x = 0$ and $y = 0$. Then, in the fiber G_1, either C_1 or C_2 is a (-1) curve. Suppose C_1 is a (-1) curve. Then we may contract C_1 without changing the Kodaira dimension (by a half-point detachment). In fact, let f: $V \to \overline{V}$ be the contraction of C_1 and let $\overline{D} = f_*(D)$. Then $f^*(\overline{D}) = D + C_1$ and $K_V = f^*(K_{\overline{V}} + C_1)$. Hence $D + K_V = f^*(\overline{D} + K_{\overline{V}})$, which implies that $\overline{\kappa}(\overline{V} - \overline{D}) = \overline{\kappa}(V - D)$. So, we may assume that the fiber G_1 leaves only one component, say A, in the affine part X. If the component A has multiplicity 1 in the fiber, there exists another (-1) curve in the fiber. Let a be the multiplicity of A. Note that the boundary divisor D now consists of M_0, M_1, G_0, G_∞ and the irreducible components of G_1 except for A. By an argument similar to the one in [15, p. 96], we then have

$$D + K_V \geq \left(1 - \frac{1}{a}\right) G_0,$$

where we note that G_0 is considered to be a general fiber of the $\mathbf{P}^1$-fibration p. If $a > 1$ this implies that $\overline{\kappa}(X) = 1$. Suppose $a = 1$. Then G_1 is already a smooth fiber and $X - C_1 \cong \mathbf{A}^1{}_* \times \mathbf{A}^1{}_*$. So, $\overline{\kappa}(X) = 0$. Q.E.D.

LEMMA 3.9. *In the case* (2) *of Theorem* 3.7, $\overline{\kappa}(X) = 1$ *provided* $b > 1$ *and* $\kappa(X) = 0$ *provided* $b = 1$.

Proof. The proof is similar to the previous case. Let $F(X, Y, Z)$ be the homogenization of the polynomial $f = x^a(x^\ell y + p(x))^b + 1$ with $x = X/Z$ and $y = Y/Z$. Consider a linear pencil L on the projective plane $\mathbf{P}^2$ spanned by the curve C defined by $F(X, Y, Z) = 0$ and $(a + (\ell + 1)b)\ell_\infty$. The pencil L has base points centered at $(1, 0, 0)$ and $(0, 1, 0)$. By eliminating the base points via the blowing-ups centered at these points, we obtain a nonsingular projective surface V and the boundary divisor D with simple normal crossings such that $X = V - D$. The proper transform of the pencil L defines on V a $\mathbf{P}^1$-fibration $p: V \to \mathbf{P}^1$. Furthermore, the divisor D contains two cross sections M_0, M_1 and two complete (possibly singular) fibers G_0, G_∞. The closures C_1, C_2 of the curves in X defined respectively by $x^\ell y + p(x) = 0$ and $x = 0$ are the irreducible components of a third singular fiber G_1 such that $C_1 \cap X \cong \mathbf{A}^1{}_*$ and $C_2 \cap X \cong \mathbf{A}^1$. By the contractions of the (-1) curves contained in the fibers G_0 and G_∞ and by the elementary transformations, we may assume that G_0 and G_∞ are the smooth fibers and that $(M_0^2) = 0$ and $(M_1^2) \leq -1$. We may also assume that C_2 is a unique (-1) curve in the fiber G_1. In fact, when there are no (-1) curves in G_1 except possibly for C_1 and C_2, the curve C_2 is a (-1) curve. By a half-point detachment and the subsequent contractions of (-1) curves in $G_1 - C_1$, we may get rid of the component C_2 and make the dual graph of G_1 a linear chain without altering the Kodaira dimension of X. The rest of the argument is the same as in Lemma 3.8. Q.E.D.

In order to treat the case (3) of Theorem 3.7, we need the following result of Fujita [6]. Let be a nonsingular projective surface and let D be a reduced effective divisor on V with normal crossings such that $V - D$ is a given affine surface X. We call such a pair (V, D) an *NC-completion* of X. We refer to [6, 8.9 and 6.21] for the definition of *NC-minimal* completion.

LEMMA 3.10. *Let X be a smooth affine rational surface with $\bar{\kappa}(X) = 0$ and* $\mathrm{Pic}(X) = 0$. *Then the following assertions hold*:

(1) *Let (V, D) be an NC-completion of X. By a sequence of blowing-downs of (-1) curves in the boundary divisor D and half-point detachments, we find an NC-minimal completion of a smooth affine surface such that X' is an open set of X and that $X - X'$ is a disjoint union of the affine lines.*

(2) *Suppose X has an NC-minimal completion. Then X is isomorphic to either one of the following two surfaces*:

(i) $X = \mathbf{P}^2 - D$, *where D is a sum of three nonconcurrent lines.*

(ii) $X = \mathbf{P}^2 - D$, *where D is a sum of a line and a smooth conic meeting each other transversally.*

(3) *Let (V, D) be an NC-completion of a smooth affine rational surface with $\bar{\kappa}(X) = 0$ and* $\mathrm{Pic}(X) = 0$. *Then the dual graph of the boundary divisor D has only one loop.*

LEMMA 3.11. *In the case* (3) *of Theorem* 3.7, *we have* $\bar{\kappa}(X) = 1$.

Proof. Note that $\deg a_0(x) > \deg a_1(x)$ by the hypothesis. Let $A_0(X, Z)$, $A_1(X, Z)$ be the homogenizations of $a_0(x), a_1(x)$, respectively, where $x = X/Z$. Let $d = \deg a_0(x)$ and $e = \deg a_1(x)$. Then the homogenization of the polynomial $f = a_0(x)y + a_1(x)$ is

$$F(X,Y,Z) = A_0(X,Z)Y + A_1(X,Z)Z^r, \qquad r := d - e + 1.$$

Write

$$a_0(x) = x^d + c_1x^{d-1} + \cdots + c_d, \qquad c_i \in k,$$

where we may assume that $a_0(x)$ is a monic polynomial. Let C be the curve defined by $F(X,Y,Z) = 0$. Then the curve C meets the line at infinity ℓ_∞ at the points $P = (1,0,0)$ and $Q = (0,1,0)$. It is clear that C meets ℓ_∞ transversally at the point P and that the defining equation of C at Q is written as

$$A_0(x, z) + A_1(x, z)z^r = 0,$$

where we denote, by abuse of notation, $X/Y, Z/Y$ by x, z, respectively. We perform the blowing-ups centered at the point Q and its infinitely near points to obtain a pair (V, D) of a smooth projective surface V and a reduced effective divisor D with simple normal crossings, which is the reduced total transform of $C + \ell_\infty$. Since $a_0(x) = 0$ has two or more roots, it follows that the dual graph of D contains more than one loop. So, $\bar{\kappa}(X) = 1$ by Lemma 3.10. Q.E.D.

Finally we prove the following:

LEMMA 3.12. *In the case* (4) *of Theorem* 3.7, *we have* $\bar{\kappa}(X) = 1$.

Proof. Let L be the linear pencil on $\mathbf{P}^2$ whose members are the closures of the curves $x^m - \lambda y^n = 0$, where $\lambda \in k \cup (\infty)$. Then L has two base points $P_0 = (0,0,1)$ and P_1, which is $(0,1,0)$ if $m > n$ and $(1,0,0)$ if $m < n$. Now, by the elimination of the base points of L, we obtain a nonsingular projective surface V on which the proper transform L' of L defines a $\mathbf{P}^1$-fibration p: $V \to \mathbf{P}^1$. The last (-1) curves arising from the elimination process of the base points give rise to two cross sections of the fibration p. Furthermore, the fibration p has two singular fibers G_1 and G_2 containing respectively the closures of the curves $x = 0$ and $y = 0$ as the irreducible components C_1 and C_2 of multiplicities m and n. We may assume that C_1 (resp. C_2) is a unique (-1) curve in G_1 (resp. G_2). Then, by virtue of [15, p. 81], we have

$$D + K_V \geq \left(1 + \left(1 - \frac{1}{m}\right) + \left(1 - \frac{1}{n}\right) - 2\right)G,$$

where D is the boundary divisor and G is a smooth complete fiber of p. Since $m > 1$, $n > 1$, and $\gcd(m, n) = 1$, we conclude $\bar{\kappa}(X) = 1$. Q.E.D.

4. DETERMINATION OF ÉTALE ENDOMORPHISMS

Let α: $X \to X$ be a nontrivial étale endomorphism. Namely we assume that α is not an automorphism. We shall then determine the k-algebra endomorphism α^*: $k[x, y, 1/f] \to k[x, y, 1/f]$ associated with α. In order to decide whether or not an étale endomorphism α of X is a finite morphism, we shall use the following well-known results.

LEMMA 4.1. *Let* α: $X \to X$ *be an étale finite endomorphism and let* $d = \deg \alpha$. *Then the Euler number* $E(X)$ *must be zero provided* $d > 1$.

Proof. Let $E(X)$ be the topological Euler number of X. Then we have $E(X) = dE(X)$, whence $E(X) = 0$ if $d > 1$. Q.E.D.

The next result is due to Suzuki [17].

LEMMA 4.2. *Let* X *be a smooth affine surface and let* ρ: $X \to B$ *be a morphism onto a smooth curve* B *whose general fibers are irreducible and* *reduced. Then we have the equality of Euler numbers*

$$E(X) = E(B)E(F) + \sum_i (E(F_i) - E(F)),$$

where F is a general fiber of ρ *and the summation is over all the singular fibers* F_i *of* ρ. *Furthermore,*

$$E(F_i) \geq E(F)$$

for all i and the equality occurs if and only if either $F \cong \mathbf{A}^1$ *or* $F \cong \mathbf{A}^1{}_*$ *and* $(F_i)_{\mathrm{red}} \cong F$.

We shall start with the case (1) of Theorem 3.7.

THEOREM 4.3. *Let* $f = x^a y^b + 1$ *with* $a \geq 1$, $b \geq 1$, $ab \neq 1$ *and* $\gcd(a, b) = 1$. *Then we have either* $a = 1$ *or* $b = 1$. *Suppose* $a = 1$ *by switching the roles of x and y if* $b = 1$. *Then* α^* *is written as*

$$\alpha^*(x) = r(xy^b + 1)^u \frac{(xy^b + 1)^m - 1}{y^b}, \qquad \alpha^*(y) = sy(xy^b + 1)^v,$$

where $r, s \in k^*$ *and* $u, v, m \in \mathbf{Z}$ *satisfying one of the following conditions*:

(i) $rs^b = 1$, $u + bv = 0$, $c = 1$, $w = m$;

(ii) $rs^b = -1$, $u + bv = -m$, $c = 1$, $w = -m$.

Let d be the degree of the endomorphism α. *Then* $d \leq m \gcd(m, v)$. *Furthermore,* α *is not a finite morphism.*

Proof. Let ρ: $X \to B$ be the $\mathbf{A}^1{}_*$-fibration whose general fibers F are defined by $x^a y^b + \lambda = 0$ with $\lambda \in k$. By Lemma 3.2, the endomorphism α induces an endomorphism β: $B \to B$ such that $\rho \circ \alpha = \beta \circ \rho$. Since $B \cong \mathbf{A}^1{}_*$, the endomorphism β is an étale finite endomorphism. Indeed, write $B = \operatorname{Spec} k[t, t^{-1}]$. Then the associated k-algebra homomorphism β^*: $k[t,t^{-1}] \to k[t, t^{-1}]$ is given as $\beta^*(t) = t^d$ after a suitable change of the coordinate t. Furthermore, there exists a splitting

$$\alpha: X \xrightarrow{\alpha'} X \times_B (B, \beta) \xrightarrow{\beta_X} X,$$

where β_X is the projection which is an étale finite morphism and α' is an étale morphism. Consider first the case $\alpha > 1$ and $b > 1$. The fibration ρ has a unique singular fiber $F_0 = aC_1 + bC_2$ with $C_1 \cong C_2 \cong \mathbf{A}^1$ and C_1, C_2 meeting transversally in a single point Q. Let $P = \rho(Q)$. Since each point of $\beta^{-1}(P)$ corresponds to a singular fiber of ρ, it follows that β is an automorphism. Hence $\alpha = \alpha'$ in the above splitting. Then the inverse image $\alpha^*(F_0)$ is supported by the fiber F_0 on the upper surface X. Note that the image $\alpha(F_0)$ is surjective onto $(F_0)_{\text{red}}$, C_1 or C_2. This implies that the point Q lies in the image $\alpha(X)$, which is an open set of X. So, $\alpha(F_0)$ is an open set of F_0. If $\alpha(F_0) \neq F_0$ there would be nonconstant invertible functions either on C_1 or C_2. This is not the case. Hence the set-theoretic image of F_0 by α is F_0, and it follows that $\alpha^*(F_0) = F_0$. Since $\gcd(a, b) = 1$, we have $\alpha^*(C_1) = C_1$ and $\alpha^*(C_2) = C_2$. So, we may write

$$\alpha^*(x) = r\left(x^a y^b + 1\right)^u x, \qquad \alpha^*(y) = s\left(x^a y^b + 1\right)^v y, \tag{1}$$

where $r, s \in k^*$ and $u, v \in \mathbf{Z}$. Since α^* sends the invertible elements of $A = k[x, y, f^{-1}]$ into the invertible elements of A, we have

$$\alpha^*\left(x^a y^b + 1\right) = c\left(x^a y^b + 1\right)^w. \tag{2}$$

Hence, by plugging the expression (1) into (2), we obtain

$$r^a s^b\left(x^a y^b + 1\right)^{au+bv} x^a y^b + 1 = c\left(x^a y^b + 1\right)^w. \tag{3}$$

By replacing $x^a y^b$ by a variable t, one may view the equation (3) as a fractional relation in t:

$$r^a s^b(t + 1)^{au+bv} t + 1 = c(t + 1)^w.$$

Then one can show easily that one of the following cases occurs:

(i) $r^a s^b = 1$, $au + bv = 0$, $c = 1$, $w = 1$;

(ii) $r^a s^b = -1$, $ua + vb = -1$, $c = 1$, $w = -1$.

Namely we have

$$\alpha^*\left(x^a y^b + 1\right) = \left(x^a y^b + 1\right)^{\pm 1}.$$

Then it is readily verified that α^* (hence α) is an automorphism.

We consider next the case $a = 1$ and $b > 1$. The other case $a > 1$ and $b = 1$ is handled in a similar fashion. Since $\beta \colon B \to B$ is a cyclic étale covering of degree, say m, the cyclic group $\omega_m = \{\zeta^i;\ i = 0, \dots, m-1\}$ acts on B so that the quotient B/ω_m is isomorphic to the lower curve B, where ζ is a primitive mth root of the unity. Let $t = xy^b + 1$. Since the coordinate ring of B is $k[t, t^{-1}]$, the action of ω is given as

$$\zeta^i \colon t \mapsto \zeta^i t.$$

Let $\beta^{-1}(P) = \{P^{(1)}, \dots, P^{(m)}\}$. Then $\beta_X^*(F_0) = F_0^{(1)} + \cdots + F_0^{(m)}$, where $F^{(i)}$ is a copy of F_0 and lies over the point $P^{(i)}$. Write $F^{(i)} = C_1^{(i)} + bC_2^{(i)}$

for $1 \le i \le m$. We may assume that $\alpha'(F_0) = F_0^{(1)}$ and $\alpha'(F_i) = C_1^{(i),*} := C_1^{(i)} \setminus (C_1^{(i)} \cap C_2^{(i)})$ for $2 \le i \le m$, where $F_i = \rho^{-1}(P^{(i)})$. The restriction of α' on F_0 is an isomorphism on each irreducible component because an étale endomorphism of $\mathbf{A}^1$ is an automorphism. Then we may write

$$\begin{aligned}\alpha^*(x) &= r'\left(xy^b + 1\right)^u x \prod_{i=1}^{m-1} \left(\zeta^i t - 1\right)\\ &= r\left(xy^b + 1\right)^u x \sum_{i=1}^{m-1} \left(t - \zeta^i\right)\\ &= r\left(xy^b + 1\right)^u x \frac{t^m - 1}{t - 1}\\ &= r\left(xy^b + 1\right)^u \frac{\left(xy^b + 1\right)^m - 1}{y^b}\end{aligned}$$

and

$$\alpha^*(y) = sy\left(xy^b + 1\right)^v,$$

where $r, r', s \in k^*$ and $u, v \in \mathbf{Z}$. Since

$$\alpha^*\left(xy^b + 1\right) = c\left(xy^b + 1\right)^w \qquad \text{with } c \in k^*,$$

we conclude that one of the following cases occurs:

(i) $rs^b = 1$, $u + bv = 0$, $c = 1$, $w = m$;

(ii) $rs^b = -1$, $u + bv = -m$, $c = 1$, $w = -m$.

So, we have an expression as given in the statement above. We note that the Jacobian determinant of $\alpha^*(x)$ and $\alpha^*(y)$ with respect to x and y is

$$rsm\left(xy^b + 1\right)^{u+v+m-1}$$

in the case (i) and

$$rsm\left(xy^b + 1\right)^{u+v-1}$$

in the case (ii), which is invertible. Let $K_0 = k(\alpha^*(x), \alpha^*(y))$. Then $(xy^b + 1)^m \in K_0$ and hence $y^m \in K_0$ because $\alpha^*(xy^b + 1) = (xy^b + 1)^w$ and $\alpha^*(y) = sy(xy^b + 1)^v$. Since the equation

$$\alpha^*(x) = r\left(xy^b + 1\right)^u \frac{\left(xy^b + 1\right)^m - 1}{y^b}$$

yields only a relation which follows from $y(xy^b + 1)^v \in K_0$, we know that $k(x, y) = K_0(y)(xy^b + 1)$ and $(xy^b + 1)^v, (xy^b + 1)^m \in K_0(y)$. Hence the degree d of the endomorphism α is less than or equal to $m \gcd(m, v)$ in both cases.

We shall show that α is not a finite morphism. Note that a general fiber of ρ is isomorphic to $\mathbf{A}^1{}_*$. More precisely, every fiber but one reducible fiber is isomorphic to $\mathbf{A}^1{}_*$ if taken with the reduced structure, and the reducible fiber consists of two affine lines meeting in one point transversally. Hence $E(X) = 1$ by Lemma 4.2. Now α must be an automorphism by Lemma 4.1 provided α is a finite morphism. Q.E.D.

COROLLARY 4.4. *Let* $X = \operatorname{Spec} k[x, y, f^{-1}]$ *with* $f = x^a y^b + 1$ *and* $a > 1$ *and* $b > 1$. *Then any étale endomorphism of* X *is an isomorphism.*

We need to treat the case $f = xy + 1$ in a different way.

THEOREM 4.5. *Let* $f = xy + 1$. *Then an étale endomorphism* α^* *is given by*

$$\alpha^*(x) = r(xy + 1)^n xg(x, y), \qquad \alpha^*(y) = s(xy + 1)^{-n} yh(x, y),$$

where $r, s \in k^*$, $rs = 1$, $n \in \mathbf{Z}$, $gh = [(xy + 1)^\ell - 1]/xy$, *and* $\ell \in \mathbf{Z}$ *or*

$$\alpha^*(x) = r(xy + 1)^n yg(x, y), \qquad \alpha^*(y) = s(xy + 1)^{-n} xh(x, y)$$

where $r, s \in k^*$, $rs = 1$, $n \in \mathbf{Z}$, $gh = [(xy + 1)^\ell - 1]/xy$, *and* $\ell \in \mathbf{Z}$. *Furthermore*, α *is not a finite morphism.*

Proof. We may write

$$\alpha^*(x) = \frac{\varphi(x, y)}{(xy + 1)^d}, \qquad \alpha^*(y) = \frac{\psi(x, y)}{(xy + 1)^e},$$

where $d, e \in \mathbf{Z}$, $\varphi(x, y)$, $\psi(x, y) \in k[x, y]$, and $d, e \geq 0$. Since $xy + 1$ is an invertible element of A, we have the relation

$$\alpha^*(xy + 1) = \gamma(xy + 1)^\ell,$$

where $\gamma \in k^*$ and $\ell \in \mathbf{Z}$. Then it follows that

$$\varphi\psi = (xy + 1)^{d+e}\left(\gamma(xy + 1)^\ell - 1\right). \tag{$*$}$$

Note that α^* is an automorphism if $\gamma = \ell = 1$. Let $t = xy + 1$. Considering the possible decomposition of the right side of the equation $(*)$ in $k[x, y]$ and switching the roles of x and y if necessary, we have the following four cases:

$$(1) \qquad \varphi = \gamma_1 t^{d'} \prod_{i=1}^{u} (t+s_i)^{\ell_i}, \qquad \psi = \gamma_2 t^{e'} \prod_{i=u+1}^{v} (t+s_i)^{\ell_i},$$

where $\gamma \neq 1$, $\gamma_1, \gamma_2 \in k^*$, $\gamma_1\gamma_2 = \gamma$, $d', e' \in \mathbf{Z}_{\geq 0}$, $d' + e' = d + e$, $s_i \in k^*$, $\prod_{i=1}^{v}(t+s_i)^{\ell_i} = t^{\ell} - \gamma^{-1}$;

$$(2) \qquad \varphi = \gamma_1 t^{d+n} xy \prod_{i=1}^{u'} (t+s_i'), \qquad \psi = \gamma_2 t^{e-n} \prod_{i=u'+1}^{v'} (t+s_i'),$$

where $n \in \mathbf{Z}$ with $d+n \geq 0$ and $e-n \geq 0$, $\prod_{i=1}^{v'}(t+s_i') = (t^{\ell}-1)/(t-1)$ with $v' = \ell - 1$ and $\gamma_1\gamma_2 = 1$;

$$(3) \qquad \varphi = \gamma_1 t^{d+n} x \prod_{i=1}^{u'} (t+s_i'), \qquad \psi = \gamma_2 t^{e-n} y \prod_{i=u'+1}^{v'} (t+s_i'),$$

where $n \in \mathbf{Z}$ with $d+n \geq 0$ and $e-n \geq 0$, $\prod_{i=1}^{v'}(t+s_i') = (t^{\ell}-1)/(t-1)$ with $v' = \ell - 1$ and $\gamma_1\gamma_2 = 1$;

$$(4) \qquad \varphi = \gamma_1 t^{d+n} y \prod_{i=1}^{u'} (t+s_i'), \qquad \psi = \gamma_2 t^{e-n} x \prod_{i=u'+1}^{v'} (t+s_i'),$$

where $n \in \mathbf{Z}$ with $d+n \geq 0$ and $e-n \geq 0$, $\prod_{i=1}^{v'}(t+s_i') = (t^{\ell}-1)/(t-1)$ with $v' = \ell - 1$ and and $\gamma_1\gamma_2 = 1$.

In the cases (1) and (2), it is easily verified that the Jacobian determinant $J(x, y)$ is 0. So, α is not an étale endomorphism.

Consider the case (3). Then, taking the differentials, we have the matrix relation

$$\begin{pmatrix} d\alpha^*(x) \\ d\alpha^*(y) \end{pmatrix} = \begin{pmatrix} \gamma_1 t^{n-1} & 0 \\ 0 & \gamma_2 t^{-n-1} \end{pmatrix} A \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where

$$A = \begin{pmatrix} (nt+t-n)g(t) + (t^2-t)g'(t) & nx^2 g(t) + x^2 t g'(t) \\ -ny^2 h(t) + y^2 t h'(t) & (t-nt+n)h(t) + (t^2-t)h'(t) \end{pmatrix}$$

and

$$g(t) = \prod_{i=1}^{u'} (t+s_i'), \qquad h(t) = \prod_{i=u'+1}^{v'} (t+s_i').$$

Now we compute $J(x, y)$ to be $\ell t^{\ell-1}$ which is an invertible element of $A = k[x, y, t^{-1}]$. This relation holds for any partition of the linear factors of $(t^{\ell} - 1)/(t - 1)$ into g and h.

We consider the case (4). Then we have the differential matrix relation

$$\begin{pmatrix} d\alpha^*(x) \\ d\alpha^*(y) \end{pmatrix} = \begin{pmatrix} \gamma_1 t^{n-1} & 0 \\ 0 & \gamma_2 t^{-n-1} \end{pmatrix} A \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where

$$A = \begin{pmatrix} ny^2 g(t) + y^2 tg'(t) & (nt + t - n)g(t) + t(t-1)g'(t) \\ (t - nt + n)h(t) + t(t-1)h'(t) & -nx^2 h(t) + x^2 th'(t) \end{pmatrix}$$

with $g(t)$ and $h(t)$ being the same as in the case (3). Then we compute $J(x, y)$ to be $-\ell t^{\ell-1}$, which is an invertible element of A. So, this case also holds for any partition of the linear factors of $(t^{\ell} - 1)/(t - 1)$ into g and h. The last assertion that α is not a finite morphism can be proved by the same argument as in Theorem 4.3. Q.E.D.

We consider next the case (2) of Theorem 3.7.

THEOREM 4.6. *Let $f = x^a(x^{\ell}y + p(x))^b + 1$ with $a, b, l > 0$, $\gcd(a, b) = 1$, $\deg p(x) < \ell$ and $p(0) \neq 0$. Then $b = 1$ and α^* is given as*

$$\alpha^*(x) = r(x^a z + 1)^u x, \qquad \alpha^*(z) = s(x^a z + 1)^v \frac{(x^a z + 1)^m - 1}{x^a},$$

where $z = x^{\ell}y + p(x)$, $r, s \in k^*$, $u, v, m \in \mathbf{Z}$, and $m \geq 2$ such that $r^a s = 1$, $au + v = 0$ or $r^a s = -1$, $au + v = -m$. Let d be the degree of the endomorphism α. Then $d \leq m \gcd(m, u)$. Furthermore, α is not a finite morphism.

Proof. We consider the $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$, which has a unique singular fiber F_0, where $F_0 = aC_1 + bC_2$ with C_1 and C_2 defined by $x = 0$ and $x^{\ell}y + p(x) = 0$, respectively. Note that $C_1 \cong \mathbf{A}^1$, $C_2 \cong \mathbf{A}^1{}_*$, and $C_1 \cap C_2 = \emptyset$. Since the $\mathbf{A}^1{}_*$-fibration ρ is defined by f, the endomorphism α of X induces an étale finite endomorphism β of B such that $\rho \circ \alpha = \beta \circ \rho$ (cf. Lemma 3.2).

Suppose first that $a > 1$ and $b > 1$. The given nontrivial étale endomorphism $\alpha\colon X \to X$ decomposes as

$$X \xrightarrow{\alpha'} X \times_B (B, \beta) \xrightarrow{\beta_X} X,$$

where α' is étale and β_X is the base change of β. Let m be the degree of β. Then the base change $\rho_B\colon X \times_B (B, \beta) \to B$ has singular fibers

$F_0^{(1)}, \ldots, F_0^{(m)}$, each of which is isomorphic to the fiber F_0. Since the image $\alpha'(X)$ in $X \times_B (B, \beta)$ is an open set and meets each of $F_0^{(1)}, \ldots, F_0^{(m)}$, there would be the singular fibers $\alpha'^{-1}(F_0^{(1)}), \ldots, \alpha'^{-1}(F_0^{(m)})$. This implies that $m = 1$. Namely, β is an automorphism. Let

$$z = x^{\ell} y + p(x) \qquad \text{and} \qquad t = x^a z^b + 1.$$

Note that the curves C_1 and C_2 are defined by $x = 0$ and $z = 0$, respectively and that t is invertible. As in the proof of Theorem 4.3, we have $\alpha(C_1) = C_1$ and $\alpha(C_2) = C_2$. Since $\gcd(a, b) = 1$, this remark implies that

$$\alpha^*(x) = rt^u x, \qquad \alpha^*(z) = st^v z,$$

where $r, s \in k^*$ and $u, v \in \mathbf{Z}$. Furthermore, we have

$$\alpha^*(t) = ct^w$$

with $c \in k^*$ and $w \in \mathbf{Z}$. As in the proof of Theorem 4.3, we deduce from these data that one of the following cases occurs:

(i) $r^a s^b = 1$, $au + bv = 0$, $c = 1$, $w = 1$;

(ii) $r^a s^b = -1$, $ua + vb = -1$, $c = 1$, $w = -1$.

Then it follows that $\alpha^*(t) = t^{\pm 1}$ and x, y are rational functions of $\alpha^*(x), \alpha^*(y)$. Namely, α is a birational étale endomorphism. So, α sends X to an open set of X injectively by Zariski's main theorem. Then α is an automorphism by [2, 4] and [5, Chap. IV, (17.9.6)].

Next consider the case where $a = 1$ and $b > 1$. With the above notations, we then have $\alpha'(F_0) = F_0^{(1)}$ and $\alpha'(F_i) = C_1^{(i),*}$ $(2 \le i \le m)$, where F_i is the fiber of ρ lying over the point $P^{(i)} := \rho_X(F_0^{(i)})$, $F_0^{(i)} = C_1^{(i)} + bC_2^{(i)}$, $C_1^{(i)} \cong \mathbf{A}^1$ and $C_1^{(i),*}$ is the $C_1^{(i)}$ minus one point. Note that the fiber F_0 is defined by $t = 1$ and $\beta\colon B \to B$ is a cyclic covering of degree m. Hence we may write

$$\alpha^*(x) = r' t^u x \prod_{i=1}^{m-1} (\zeta^i t - 1) = rt^u x \frac{t^m - 1}{t - 1} \qquad \text{and} \qquad \alpha^*(z) = st^v z,$$

where $r', r, s \in k^*$ and $u, v \in \mathbf{Z}$ (cf. the proof of Theorem 4.3). Furthermore, we have

$$\alpha^*(t) = ct^w$$

with $c \in k^*$ and $t \in \mathbf{Z}$. As in the proof of Theorem 4.3, we deduce from these data that one of the following cases takes place:

(i) $rs^b = 1$, $u = bn$, $v = -n$, $c = 1$, $w = m$;

(ii) $rs^b = -1$, $u + bv = -m$, $c = 1$, $w = -m$.

We denote the Jacobian matrix of $\alpha^*(x)$ and $\alpha^*(y)$ with respect to x and y by $J(\alpha^*(x), \alpha^*(y)/x, y)$. The relation

$$\begin{pmatrix} dx \\ dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & x^{\ell} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

and the chain rule imply the following matrix relation:

$$\begin{pmatrix} d\alpha^*(x) \\ d\alpha^*(y) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & \alpha^*(x)^{\ell} \end{pmatrix}^{-1} J\left(\frac{\alpha^*(x), \alpha^*(z)}{x, z}\right) \begin{pmatrix} 1 & 0 \\ * & x^{\ell} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Thus $|(\alpha^*(x), \alpha^*(y)/x, y)|$ is the determinant of the matrix

$$\begin{pmatrix} 1 & 0 \\ * & \alpha^*(x)^{\ell} \end{pmatrix}^{-1} J\left(\frac{\alpha^*(x), \alpha^*(z)}{x, z}\right) \begin{pmatrix} 1 & 0 \\ * & x^{\ell} \end{pmatrix},$$

which is equal to

$$\frac{x^{\ell}}{\alpha^*(x)^{\ell}} \left| J\left(\frac{\alpha^*(x), \alpha^*(z)}{x, z}\right) \right|.$$

We denote this by J for the sake of simplicity. We obtain $|J(\alpha^*(x), \alpha^*(z)/x, z)|$ is equal to $rsmt^{u+v+m-1}$ in the case (i) and to $rsmt^{u+v-1}$ in the case (ii). So, J is invertible in A if and only if

$$\frac{\alpha^*(x)^{\ell}}{x^{\ell}} = r^{\ell} t^{u\ell} (t^{m-1} + t^{m-2} + \cdots + 1)^{\ell}$$

is an invertible element in A. This is the case only if $m = 1$. Hence $m = 1$ and α^* is an automorphism as in the previous case $a > 1$ and $b > 1$.

Consider the case $a > 1$ and $b = 1$. Let $t = x^a z + 1$. With the same notations as above, we have $\alpha'(F_0) = F_0^{(1)}$ and $\alpha'(F_i) = C_2^{(i)}$ $(2 \le i \le m)$, where $F_i = \rho^{-1}(P^{(i)})$, $F_0^{(i)} = aC_1^{(i)} + C_2^{(i)}$, $C_1^{(i)} \cong \mathbf{A}_1$, and $C_2^{(i)} \cong \mathbf{A}_1^*$. We know that $\beta\colon B \to B$ is a cyclic covering of degree m. So, we have the following expression of α^*

$$\alpha^*(x) = rt^u x$$

and

$$\alpha^*(z) = st^v z \prod_{i=2}^{m} (\zeta^{i-1} t - 1)$$

$$= st^v z \frac{t^m - 1}{t - 1},$$

where $r, s, s' \in k^*$ and $u, v \in \mathbf{Z}$. Furthermore, we have

$$\alpha^*(t) = ct^w$$

with $c \in k^*$ and $w \in \mathbf{Z}$. As in the proofs of Theorem 4.3 and the previous cases, we obtain the following two cases:

(i) $r^a s = 1,\ au + v = 0,\ c = 1,\ w = m;$

(ii) $r^a s = -1,\ au + v = -m,\ c = 1,\ w = -m.$

We have then $|J(\alpha^*(x), \alpha^*(z)/x, z)|$ is equal to $rsmt^{u+v+m-1}$ in the case (i) and to $rsmt^{u+v-1}$ in the case (ii). Since $\alpha^*(x)^\ell/x^\ell = r^\ell t^{u\ell}$, J is invertible in A. So, α^* with the above expression is an étale endomorphism. Hence the determinant J considered in the case $\alpha > 1,\ b = 1$, which is equal to

$$\frac{rsmt^N}{r^\ell}, \qquad \text{where } N = u + v + m - u\ell - 1 \text{ or } u + v - u\ell - 1,$$

is an invertible element of $A = k[x, y, 1/t]$.

Consider finally the case where $a = b = 1$. With the same notation as above, we have

$$\alpha'(F_i) = C_1^{(i),*} \qquad \text{or} \qquad \alpha'(F_i) = C_2^{(i)}$$

for $2 \le i \le m$ and

$$(1) \quad \alpha'(C_1) = C_1^{(1)} \qquad \text{and} \qquad \alpha'(C_2) = C_1^{(1),*}$$

or

$$(2) \quad \alpha'(C_1) = C_1^{(1)} \qquad \text{and} \qquad \alpha'(C_2) = C_2^{(1)}.$$

In the cases (1) and (2) above, we have, respectively,

$$\alpha^*(x) = rt^u xz(t - a_1) \cdots (t - a_p)$$
$$\alpha^*(z) = st^v(t - b_1) \cdots (t - b_q)$$

and

$$\alpha^*(x) = rt^u x(t - a_1) \cdots (t - a_p)$$
$$\alpha^*(z) = st^v z(t - b_1) \cdots (t - b_q),$$

where $r, s \in k^*$, $\{a_1, \dots, a_p, b_1, \dots, b_q\} = \{\zeta^i \mid i = 1, \dots, m-1\}$ with $p + q = m - 1$ and $u, v \in \mathbf{Z}$. Furthermore, we have

$$\alpha^*(t) = ct^w$$

with $c \in k^*$ and $w \in \mathbf{Z}$. From the last relation, we can easily deduce

(i) $rs = 1, u + v = 0, c = 1, w = m,$

(ii) $rs = -1, u + v = -m, c = 1, w = -m.$

Now compute

$$J := \frac{x^{\ell}}{\alpha^*(x)^{\ell}} \left| J\left(\frac{\alpha^*(x), \alpha^*(z)}{x, z}\right)\right|.$$

In the case (1), we have $|J(\alpha^*(x), \alpha^*(z)/x, z) = 0$, and this case does not occur. In the second case, we have

$$\begin{aligned}
&\left| J\left(\frac{\alpha^*(x), \alpha^*(z)}{x, z}\right)\right| \\
&\quad = rst^{u+v}\frac{t^m - 1}{t - 1} + rs(u + v)t^{u+v-1}(t^m - 1) \\
&\qquad + rst^{u+v}(t - 1)\left\{(t - a_1)\cdots(t - a_p)\left(\sum_{j=1}^{q}\frac{\prod_{k=1}^{q}(t - b_k)}{t - b_j}\right)\right. \\
&\qquad\qquad \left. +(t - b_1)\cdots(t - b_q)\left(\sum_{i=1}^{p}\frac{\prod_{k=1}^{p}(t - a_k)}{t - a_i}\right)\right\},
\end{aligned}$$

which is equal to mt^{m-1} in the case (i) and to $-mt^{-(m+1)}$ in the case (ii) and

$$\left(\prod_{i=1}^{p}(t - a_i)\right)^{\ell} \Bigg| \left| J\left(\frac{\alpha^*(x), \alpha^*(z)}{x, z}\right)\right|$$

because J is invertible in A. Note that $a_i^m = 1$ for $1 \le i \le p$. Then it is easily verified that the above division occurs only if the set $\{a_1, \ldots, a_p\}$ is void, i.e., $q = m - 1$. In this case the expression of α^* is the same as in the case $a > 1$ and $b = 1$. The estimation of the degree d of the étale endomorphism α is similar to the case in Theorem 4.3.

The last assertion that α is not a finite morphism follows from Lemma 4.1 if one notes that the fiber F_0 is isomorphic to $\mathbf{A}^1{}_*$. Q.E.D.

We shall consider the case (3) of Theorem 3.7. For a polynomial $f = a_0(x)y + a_1(x)$ with $\gcd(a_0(x), a_1(x)) = 1$, we note that the following assertions hold:

(1) $f - \lambda$ is reducible for $\lambda \in k$ if and only if $a_0(x)$ and $a_1(x) - \lambda$ have a common factor.

(2) A irreducible curve C defined by an equation $b_0(x)y + b_1(x) = 0$ with $b_0(x), b_1(x) \in k[x] \setminus (0)$ has only one place at infinity if and only if $b_0(x) \in k^*$ (cf. the proof of Lemma 3.11). Suppose $\deg b_1(x) < \deg b_0(x)$. Let n be the number of distinct linear factors of $b_0(x)$. Then the curve C has exactly $n + 1$ places at infinity.

(3) Since $\deg a_1(x) < \deg a_0(x)$, any irreducible divisor of $f - \lambda$ other than linear polynomials of the form $x - c$ is of the form $b_0(x)y + b_1(x)$ with $\deg b_1(x) < \deg b_0(x)$.

THEOREM 4.7. *In the case* (3) *of Theorem* 3.7, *write* $f = a_0(x)y + a_1(x)$ *with* $\gcd(a_0(x), a_1(x)) = 1$ *and* $\deg a_1(x) < \deg a_0(x)$. *Suppose that an étale endomorphism* α *of* X *preserves the* $\mathbf{A}^1{}_*$*-fibration* $\rho\colon X \to B$. *Then* α^* *is written as*

$$\alpha^*(x) = s(x - \gamma),$$

$$\alpha^*(y) = s^{-\deg a_0(x)} \frac{\delta(a_0(x)y + a_1(x))^{\ell} - a_1(s(x - \gamma))}{a_0(x)},$$

where $s, \delta \in k^*$, $\gamma \in k$, *and* $\ell \in \mathbf{Z}$ *satisfy the condition that* s *is an nth root of unity with n being the number of distinct roots of* $a_0(x) = 0$ *and*

$$a_0(x) \Big| \left(\delta a_1(x)^{\ell} - a_1(s(x - \gamma))\right).$$

Furthermore, no nontrivial étale endomorphisms α *of* X *are finite morphisms. The last assertion holds without assuming that* α *preserves the fibration* $\rho\colon X \to B$.

Proof. Our proof consists of two steps.

(I) Now look at the $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$. In this case, x is a parameter of the base curve $B \cong \mathbf{A}^1$. By the hypothesis, α preserves the $\mathbf{A}^1{}_*$-fibration ρ. Note that every fiber of ρ is reduced and isomorphic to either $\mathbf{A}^1{}_*$ or $\mathbf{A}^1$ and that the fiber F_0 of ρ over a point $x = c$ is isomorphic to $\mathbf{A}^1$ provided c is a root of $a_0(x) = 0$. Note that the endomorphism $\beta\colon B \to B$ is an automorphism because β is étale in this case and $B \cong \mathbf{A}^1$. The restriction α' on F_0 is an isomorphism onto a fiber isomorphic to $\mathbf{A}^1$. Then we write $\alpha^*(x - c) = s(x - \gamma)$, where $a_0(c) = a_0(\gamma) = 0$ and $s \in k^*$. So, we may assume $\alpha^*(x) = s(x - \gamma)$. Since

$$\alpha^*(a_0(x)y + a_1(x)) = \delta(a_0(x)y + a_1(x))^{\ell},$$

with $\delta \in k^*$ and $\ell \in \mathbf{Z}$, we conclude that

$$\alpha^*(y) = \frac{\delta(a_0(x)y + a_1(x))^{\ell} - a_1(s(x-\gamma))}{a_0(s(x-\gamma))}.$$

Taking the differentials, we have the following matrix relation:

$$\begin{pmatrix} d\alpha^*(x) \\ d\alpha^*(y) \end{pmatrix} = \begin{pmatrix} s & 0 \\ * & \dfrac{\delta\ell(a_0(x)y + a_1(x))^{\ell-1}}{a_0(s(x-\gamma))} a_0(x) \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

So, the Jacobian determinant is

$$sa_0(x)\frac{\delta\ell(a_0(x)y + a_1(x))^{\ell-1}}{a_0(s(x-\gamma))}.$$

This must be an invertible element of $A = k[x, y, (a_0(x)y + a_1(x))^{-1}]$. Hence we have $a_0(s(x-\gamma)) = da_0(x)$ with $d \in k^*$. Let $\{\alpha_1, \dots, \alpha_n\}$ be the set of distinct roots of $a_0(x) = 0$, where $n \geq 2$ by the hypothesis. Then the mapping $\alpha \mapsto \gamma + \alpha/s$ induces a permutation of the set $\{\alpha_1, \dots, \alpha_n\}$. Hence it follows that $s^n = 1$. Since $\alpha^*(y)$ is a polynomial in x and y, we must have

$$a_0(x) \mid \delta a_1(x)^{\ell} - a_1(s(x-\gamma)).$$

Then $\alpha^*(x)$ and $\alpha^*(y)$ are expressed in the form as given in the statement.

(II) In the case (3), the base curve B of the $\mathbf{A}^1{}_*$-fibration $\rho: X \to B$ is isomorphic to $\mathbf{A}^1$. Let n be the number of distinct roots of the equation $a_0(x) = 0$. If $f(c) = 0$ for $c \in k$ then the fiber of ρ over the point $x = c$ is isomorphic to $\mathbf{A}^1$. Hence the fibration ρ has as many singular fibers as the distinct roots of $a_0(x) = 0$ which are isomorphic to $\mathbf{A}^1$, and the other fibers of ρ are isomorphic to $\mathbf{A}^1{}_*$. By Lemma 4.2, we then have the Euler number $E(X) = n$. Suppose that an étale endomorphism $\alpha: X \to X$ is a finite morphism and not an automorphism. Hence α is an automorphism unless $n = 0$ by Lemma 4.1. Since $n \geq 2$, this is a contradiction. Q.E.D.

Remark. Note that $a_0(x) = 0$ has two or more roots. Look at the fibration $\varphi: X \to \mathbf{A}^1{}_*$ defined by the pencil $\{f = \lambda;\ \lambda \in k^*\}$. Let G_λ be the fiber defined by $f = \lambda$. Then G_λ is reducible if and only if $a_0(x)$ and $a_1(x)$ have a common factor $x - c$ with $c \in k$ [cf. the above remark (1) before Theorem 4.7). Then G_λ contains an affine line $x = c$. We claim that an affine line, say C, defined by $x = c$ and contained in a fiber of φ is mapped by $\alpha\colon X \to X$ to an affine line $x = d$ and contained in a fiber of φ. Indeed, if $\alpha(C)$ is not contained in a fiber of φ, there is a dominant morphism from C to $\mathbf{A}^1{}_*$. This is impossible. Hence $\alpha(C)$ is contained in a fiber of φ. Since $\alpha(C)$ has one place at infinity and since $\deg a_1(x) < \deg a_0(x)$ by the hypothesis, we conclude that $\alpha(C)$ is an affine line defined by $x = d$ for some $d \in k$.

Finally, we consider the case (4) in Theorem 3.7.

THEOREM 4.8. *Let* $X = \operatorname{Spec} k[x, y, f^{-1}]$ *with* $f = y^m - x^n$. *Then the following assertions hold*:

(1) *Any étale endomorphism of X is a finite cyclic covering.*

(2) *It is given as*

$$\alpha^*(x) = rf^u x, \qquad \alpha^*(y) = sf^v y,$$

where $r, s \in k^*$ *with* $r^n = s^m$ *and* $u, v \in \mathbf{Z}$

Proof. (1) Consider the fibration $\varphi\colon X \to \mathbf{A}^1{}_*$ given by the pencil $\{f = \lambda;\ \lambda \in k^*\}$. Let G_λ be a fiber $f = \lambda$. Then G_λ is a smooth irreducible affine curve of fixed positive geometric genus and with only one place at infinity. Then the coordinate ring of G_λ has no nonconstant invertible elements. This implies that $\alpha(G_\lambda)$ is again a fiber of φ. Namely there exists an endomorphism $\psi\colon \mathbf{A}^1{}_* \to \mathbf{A}^1{}_*$ such that $\varphi \circ \alpha = \psi \circ \varphi$. Suppose $\alpha(G_\lambda) = G_\mu$ for some $\mu \in k^*$. Since both G_λ and G_μ are curves with only one place at infinity, the restriction of $\alpha|_{G_\lambda}$ is a finite étale morphism. Since the Euler numbers of G_λ and G_μ are negative, the degree of $\alpha|_{G_\lambda}$ must be one. Namely the endomorphism α is the base change of $\psi\colon \mathbf{A}^1{}_* \to \mathbf{A}^1{}_*$ up to an automorphism of X. Since ψ is a finite cyclic covering, so is α.

(2) By lemma 3.2, α preserves the $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$. The fibration ρ has two singular fibers $F_1 = mC_1$ and $F_2 = nC_2$, where C_1 and C_2 are defined by $y = 0$ and $x = 0$, respectively. Note that $C_1 \cong C_2 \cong \mathbf{A}^1{}_*$. Since the $\mathbf{A}^1{}_*$-fibration ρ is canonical for the surface X, the endomorphism α of X induces a finite endomorphism, which we denote anew by β, of $B \cong \mathbf{A}^1$, such that $\rho \circ \alpha = \beta \circ \rho$ and β ramifies possibly over the points $P_i = \rho(F_i)$ for $i = 1, 2$. Note that $m > 1$ and $n > 1$ since $\bar{\kappa}(X) = 1$. Let $d = \deg \beta$. Then β ramifies totally over the points P_1 and P_2 because there are no multiple fibers in the fibration ρ other than F_1 and F_2. The finite endomorphism $\beta\colon B \to B$ extends to a finite morphism $\tilde{\rho}\colon \mathbf{P}^1 \to \mathbf{P}^1$, which totally ramifies at the point P_∞ at infinity and the points P_1 and P_2. Then $d = 1$ by the Riemann–Hurwitz formula. Then α^* is given in the form

$$\alpha^*(x) = rf^u x, \qquad \alpha^*(y) = sf^v y,$$

where $r, s \in k^*$ and $u, v \in \mathbf{Z}$. Note that the case $\beta(P_1) = P_2$ and $\beta(P_2) = P_1$ is impossible since $\gcd(m, n) = 1$. For otherwise, we would obtain a relation of divisors $\alpha^*(mC_1) = nC_2$, which is impossible. Furthermore, we must have

$$\alpha^*(f) = cf^w$$

with $c \in k^*$ and $w \in \mathbf{Z}$. If $mv \geq un$, this yields a relation

$$s^m f^{mv-un} y^m - r^n x^n = cf^{w-un}.$$

Hence it follows that

$$d = 1, \qquad s^m = r^n = c, \qquad mv = un, \qquad w - un = 1.$$

If $un \geq mv$, we have the same relation as above. Then a computation of the Jacobian determinant $J(x, y)$ of $\alpha^*(x)$ and $\alpha^i(y)$ with respect to x and y gives rise to

$$J(x, y) = rs(1 + mv)f,$$

where $w = 1 + mv \neq 0$ since $m > 1$. So, the morphism α given by $\alpha^*(x) = rf^u x$ and $\alpha^*(y) = sf^v y$ is an étale endomorphism of $X = \operatorname{Spec} k[x, y, f^{-1}]$. We shall show that α is a finite morphism. Let $A = k[x, y, f^{-1}]$. Since $\alpha^*(f) = cf^w$, f is integral over $\alpha^*(A)$. Meanwhile, it is easy to see that

$$\alpha^*\left(\frac{x^n}{f}\right) = \frac{x^n}{f}, \qquad \alpha^*\left(\frac{y^m}{f}\right) = \frac{y^m}{f}.$$

Hence $x^n = \alpha^*(x^n/f)f$ and $x^n/f \in A$, which implies that x is integral over $\alpha^*(A)$. Similarly, y is integral over $\alpha^*(A)$. So, A is integral over $\alpha^*(A)$, and α is a finite morphism. Q.E.D.

Remark. With the above notation, assume without loss of generality that $n > m$ and let

$$N = mn(t-1) + 1 \qquad \text{and} \qquad e = n(m - m_1)(t-1) + q_0 + 1,$$

where t, q_0, m_1 are integers such that $t \geq 2$, $n = mq_0 + m_1$ with $q_0 > 0$ and $0 < m_1 < m$. Consider a cyclic group action of order N on the affine plane $\mathbf{A}^2 = \operatorname{Spec} k[x, y]$ given by

$$x \mapsto \zeta x, \qquad y \mapsto \zeta^e y,$$

where ζ is a primitive Nth root of unity and $e = n(m - m_1)(t-1) + q_0 + 1$. Let $G = \{\zeta^i; 0 \leq i < N\}$. Then $y^m - \lambda x^n$ is a semi-invariant polynomial of weight n. The G-action on $\mathbf{A}^2$ is extended to a G-action on the projective plane $\mathbf{P}^2 = \operatorname{Proj} k[X, Y, Z]$ by setting

$$X \mapsto \zeta X, \qquad Y \mapsto \zeta^e Y, \qquad Z \mapsto Z,$$

where $x = X/Z$ and $y = Y/Z$. Consider a pencil $\Lambda = \{Y^m Z^{n-m} = \lambda X^n; \lambda \in k \cup (\infty)\}$ on $\mathbf{P}^2$. The pencil Λ has base points at $(X, Y, Z) = (0,0,1), (0,1,0)$ and their infinitely near points. One can eliminate the base points of the pencil Λ by G-equivariant blowing-ups. Namely, in each step of the blowing-ups, the center of blowing-up is a G-fixed point under the induced G-action. Let $V \to \mathbf{P}^2$ be a composite of the blowing-ups such that the proper transform of the pencil Λ is free from the base points. Then the last (-1) curves (there are two of them) appearing in the last step of the blowing-ups are contained in the fixed-point locus under the induced G-action on V. We thus obtain a $\mathbf{P}^1$-fibration $\varphi\colon V \to \mathbf{P}^1$ which has two singular fibers F_0 and F_∞ coming from the multiple members $y^m = 0$ and $x^n = 0$ of Λ and whose smooth fibers are the property transforms of the other members of the pencil Λ. Consider the quotient space $W := V/G$. Then there is a $\mathbf{P}^1$-fibration on W whose fibers are the images of the fibers of the $\mathbf{P}^1$-fibration φ under the quotient morphism $\pi\colon V \to W$. The fibers $\pi(F_0)$ and $\pi(F_\infty)$ have quotient singular points. Let $\tilde{W}$ be the minimal resolution of the quotient singularities on W. Then one can show that $\tilde{W}$ contains X as a Zariski open set and that the quotient morphism $\pi\colon V \to W$ induces a finite étale endomorphism $\alpha\colon X \to X$ of degree N. Thus one can construct geometrically the finite étale endomorphisms of X. The details are given in the Appendix.

APPENDIX:

Geometric Construction of Certain Finite Étale Endomorphisms

Hisayo Aoki and Masayoshi Miyanishi

1. *Introduction*

In Theorems 1.3 and 4.8, the following result is proved.

THEOREM. *Suppose $\bar{\kappa}(X) \geq 0$. Then X has no étale finite endomorphisms unless X is isomorphic to the surface $\mathbf{A}^2 - F_0$, where F_0 is a curve defined by $f := y^m - x^n = 0$ for positive integers m, n with $\gcd(m, n) = 1$. If $f = y^m - x^n$, any étale endomorphism $\alpha\colon X \to X$ is a finite cyclic covering and has the following expression in terms of the associated algebra endomorphism α^* of $k[x, y, f^{-1}]$*

$$\alpha^*(x) = rf^u x, \qquad \alpha^*(y) = sf^v y,$$

where $r, s \in k^*$ *with* $r^n = s^m$ *and* $u, v \in \mathbf{Z}$. *Furthermore*, $\deg \alpha = 1 + un = 1 + vm$.

The purpose of the present Appendix is to give a geometric construction of the above finite cyclic covering for the surface $X = \mathbf{A}^2 - F_0$, where F_0 is defined by $f := y^m - x^n = 0$. A main result of the Appendix is Theorem 3.2 below and the crucial result is Lemma 3.1. By Theorem 3.12 in the main body, X has the logarithmic Kodaira dimension one and that X has an $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ with $B \cong \mathbf{A}^1$, which is given by a linear pencil $\Lambda = \{y^m - \lambda x^n = 0;\ \lambda \in \mathbf{P}^1\}$ with one member $y^m = x^n$ removed. Furthermore, an étale endomorphism $\alpha\colon X \to X$ preserves the $\mathbf{A}^1{}_*$-fibration by Lemmas 3.2 and 3.3. In other terms, there exists an endomorphism β: $B \to B$ such that $\rho \circ \alpha = \beta \circ \rho$. Throughout the appendix, we assume that $m < n$ without loss of generality.

2. *Action of a Cyclic Group on X*

Let $\alpha\colon X \to X$ be an étale finite endomorphism given as above. The proof of Theorem 4.8 (1) implies that X is isomorphic to $X \times_B (B, \beta)$ and the endomorphism α is obtained as the base change β_X. Hence we have:

LEMMA 2.1. *With the notations in the above theorem*, α *is a finite cyclic covering of degree* $N := 1 + un = 1 + vm$. *Let* $G = \{g^i;\ 0 \le i < N\}$ *be a cyclic group of order* N *and let* ζ *be a primitive* N*th root of unity. Then the* G*-action on* X *is given as*

$$^g x = \zeta^{-u} x, \qquad ^g y = \zeta^{-v} y.$$

Let $Y = \mathbf{A}^2 - \{0\}$. Then $X = \operatorname{Spec} k[x, y, f^{-1}]$ is a Zariski open set of Y. In fact, $X = Y - F_0$, where F_0 is the curve defined by $f = 0$. There is an $\mathbf{A}^1{}_*$-fibration $\tilde{\rho}\colon Y \to \tilde{B}$, where $\tilde{B} \cong \mathbf{P}^1$, and the $\mathbf{A}^1{}_*$-fibration $\rho\colon X \to B$ is the restriction of $\tilde{\rho}$ onto the open set X. Furthermore, the $\mathbf{A}^1{}_*$-fibration $\tilde{\rho}$ has two multiple irreducible fibers with respective multiplicities m and n. The endomorphism $\beta\colon B \to B$ such that $\rho \circ \alpha = \beta \circ \rho$ extends to an endomorphism $\tilde{\beta}\colon \tilde{B} \to \tilde{B}$. We denote the point $\tilde{B} - B$ by P_∞. Then $\tilde{\beta}^{-1}(P_\infty)$ consists of a single point P_∞. Hence $\tilde{\beta}$ is totally ramified over P_∞. We shall look into the ramification of the morphism $\tilde{\beta}$ on B. Note that $\tilde{\beta}^{-1}(B) = \beta^{-1}(B) = B$.

LEMMA 2.2. *Let the notation and the assumptions be the same as above. Then the following assertions hold*:

(1) *Let* P *be a point of* B *such that the fiber of* ρ *over* P *is reduced. Then* β *is unramified over* P.

(2) *Let* P_0 *and* P_1 *be the points of* B *such that the fibers of* ρ *over* P_0 *and* P_1 *are multiple fibers of respective multiplicities* m *and* n. *Then* $\beta^{-1}(P_0)$

(*resp.* $\beta^{-1}(P_1)$) *consists of one unramified point and points* $\tilde{P}_0^{(1)}, \ldots, \tilde{P}_0^{(p)}$ *with ramification index* m (*resp. one unramified point and points* $\tilde{P}_1^{(1)}, \ldots, \tilde{P}_1^{(q)}$ *with ramification index* n).

Proof. (1) Let t be a local parameter of B at the point P. Let $\tilde{P}$ be a point of B which maps to P. Suppose $\beta^*(t) \sim \tau^a$ at the point $\tilde{P}$ with $a > 1$. Let R be a smooth point of the fiber $2\rho^{-1}(P)$ and let u be a local parameter of the curve $\rho^{-1}(P)$ at R .Then $\{t, u\}$ is a local system of parameters of X at the point R. Let $\tilde{R}$ be a point on the fiber of ρ over the point $\tau = 0$ such that $\beta(\tilde{R}) = R$. Since $\alpha\colon X \to X$ is unramified at $\tilde{R}$, the set $\{\beta^*(t), \beta^*(u)\}$ is a local system of parameters, while this is not the case because $\beta^*(t) \sim \tau^a$. This is a contradiction. Hence β is unramified over P.

(2) Let Q be a point of B which maps to P_0. Suppose that β ramifies at the point Q with ramification index e. Take a local parameter t of B at the point P_0 and a local parameter τ at the point Q. Then $\beta^*(t) > \tau^e$. Write $\rho^*(P_0) = m\Gamma_0$ with $\Gamma_0 \cong \mathbf{A}^1{}_*$. Take a smooth point R on Γ_0 and a local parameter u of Γ_0 at the point R. Then $t \sim u^m$. In the subsequent arguments, we denote $\beta^*(u)$ by the same letter u for the sake of simplifying the notation. Take a point $\tilde{R}$ of the upper X which maps to Q by ρ and R by α. Then, near the point, we have $\tau^e \sim u^m$, where we note again that $\tau^e \sim u^m$ is synonymous to $\tau^e/u^m = \sigma$ with $\sigma(\tilde{R}) \neq 0$. Let $\ell = \gcd(e, m)$, and write $e = e'\ell$ and $m = m'\ell$. Since X is normal at $\tilde{R}$ the relation $(\tau^{e'}/u^{m'})^{\ell} = \sigma$ implies that $\sigma' = \tau^{e'}/u^{m'}$ is an invertible element of the local ring $\mathscr{O}_{X, \tilde{R}}$. Then the Euclidean algorithm implies that there exists an element v of $\mathscr{O}_{X, \tilde{R}}$ such that $\tau \sim v^{m'}$ and $u \sim v^{e'}$. Since u forms a local system of parameters at R with another element and since α is étale, it follows that $e' = 1$. Hence $e = \ell$ and $m = m'e$. Suppose that $m' > 1$. Then the fiber $\rho^{-1}(Q)$ is a multiple fiber. Hence its multiplicity is either m or n. Since $u \sim v$ and $\tau \sim v^{m'}$, it follows that m' is the multiplicity and $= m'$. In this case $e = 1$. If $m' = 1$. Then $e = m$. This yields the assertion for the point P_0. The argument is the same for the point P_1. Q.E.D.

A consequence of the above lemma is the following:

COROLLARY 2.3. *Let $\beta\colon B \to B$ be the endomorphism defined as above. Then β is an automorphism.*

Proof. Let δ be the degree of β. Since $\tilde{\beta}$ is totally unramified over the point P_∞, by the Riemann–Hurwitz formula, we have

$$1 + mp = \delta = 1 + nq, \tag{1}$$

$$-2 = -2\delta + (\delta - 1) + (m-1)p + (n-1)q. \tag{2}$$

By (1) we have $mr = ns$ and by (2) we have

$$\delta - 1 = mp + nq - (p + q).$$

Replacing δ by $\delta = 1 + nq$, we have $q = (m-1)p$. Similarly, we have $p = (n-1)q$. Hence

$$q = (m-1)(n-1)q.$$

Since $m > 1$ and $n > 1$, this implies $q = 0$. Similarly, $p = 0$. This implies $\delta = 1$. Hence β is an automorphism. Q.E.D.

Let K be the function field of B over k and let X_K be the generic fiber of ρ, i.e., $X_K = X \times_B \operatorname{Spec} K$. Then $X_K \cong \mathbf{A}^1{}_{*,K} = \operatorname{Spec} k[\xi, \xi^{-1}]$ because the $\mathbf{A}^1{}_*$-fibration ρ is untwisted. Set

$$t = \frac{y^m}{y^m - x^n}.$$

Then $B = \operatorname{Spec} k[T]$ and $K = k(t)$. The generic fiber X_K is the affine curve

$$y^m = \frac{t}{t-1} x^n \tag{3}$$

on $\mathbf{A}^2_K$ with the point $(x, y) = (0, 0)$ deleted. Normalizing the curve (3) by taking the fractions of the powers of x and y, we may assume that $\xi = y^a / x^b$ and

$$y = \left(\frac{t}{t-1}\right)^c \xi^n, \qquad x = \left(\frac{t}{t-1}\right)^d \xi^m$$

for some integers a, b, c, d. On the other hand, the étale endomorphism $\alpha\colon X \to X$ induces an étale finite endomorphism $\alpha_K\colon X_K \to X_K$, which must be a cyclic Galois covering. Indeed, $\alpha_K^*(\xi) = \xi^N$, where N is the degree of α. The cyclic group $G = \{g^i;\ 0 \le i < N\}$ of order N acts on X_K and the quotient X_K/G is the lower X_K. With respect to the G-action, the element t is invariant. Since $un = vm$ and $\gcd(m, n) = 1$, we may write $u = m\ell$ and $v = n\ell$ for some integer ℓ. Since $na = bm + 1$, it follows that ${}^g\xi = \zeta^{-\ell}\xi$.

So, the G-action is given as follows:

$${}^g\xi = \zeta^{-\ell}\xi, \qquad {}^g x = \zeta^{-u} x, \qquad {}^g y = \zeta^{-v} y.$$

Let $X' + X/G$ and let $\pi\colon X \to X'$ be the quotient morphism. Then $\alpha\colon X \to X$ factors as

$$\alpha\colon X \xrightarrow{\pi} X' \xrightarrow{\alpha'} X.$$

Since the G-action on X is free on the open set $\{x \neq 0,\ y \neq 0\}$, the degree of π is N and hence $\alpha' \colon X' \to X$ is a birational finite morphism. Since X is smooth and X' is normal, α' is an isomorphism by Zariski's main theorem. Thus we also see that $\alpha \colon X \to X$ is a cyclic Galois covering with group G and the G-action on the upper X extends to the affine plane $\mathbf{A}_k^2$. Note that

$$u = m\ell, \qquad v = n\ell, \qquad N = 1 + mn\ell.$$

Since $\gcd(N, \ell) = 1$, $\zeta^{-\ell}$ is a primitive Nth root of unity. If one replaces ζ by $\zeta^{-\ell}$ then the G-action is normalized is such a way that

$$^{g}\xi = \zeta\xi, \qquad ^{g}x = \zeta^{m}x, \qquad ^{g}y = \zeta^{n}y, \qquad ^{g}f = \zeta^{mn}f. \tag{4}$$

Replacing in (4) a primitive Nth root ζ by ζ^{m}, we may further normalize the G-action as

$$^{g}x = \zeta x, \qquad ^{g}y = \zeta^{e}y, \qquad ^{g}\xi = \zeta^{-n\ell}\xi, \qquad ^{g}f = \zeta^{n}f, \tag{5}$$

where e is the unique integer such that $0 < e < d$ and $em \equiv n \pmod{N}$. We use this expression in order to look into the cyclic singularity on X/G.

We define the integers $q_0, q_1, \ldots, q_s$ by the following Euclidean algorithm applied to the given positive integers m, n with $m < n$ and $\gcd(m, n) = 1$

$$\begin{array}{rlc}
n &= q_0 m_0 + m_1, & 0 < m_1 < m_0, \\
m_0 &= q_1 m_1 + m_2, & 0 < m_1 < m_2 \\
\cdots & \quad \cdots & \cdots \\
m_{s-2} &= q_{s-1} m_{s-1} + m_s, & 0 < m_{s-1} < m_s \\
m_{s-1} &= q_s m_s, & m_s = 1,
\end{array}$$

where $m_0 = m$. Consider a linear pencil $\Lambda = \{y^m = \lambda x^n\}_{\lambda \in \mathbf{P}^1}$, where the member is $x^n = 0$ if $\lambda = \infty$. The pencil Λ has base points at the point $(x, y) = (0, 0)$ and its infinitely near points. The blowing-ups with centers at these base points are G-equivalent blowing-ups and will give exceptional curves whose weighted dual graph G_s is given as

$$\begin{tikzcd}[arrows=dash]
{[q_0-1]} \arrow[r] & \overset{-(q_1+2)}{\circ} \arrow[r] & {[q_2-1]} \arrow[r] & \overset{-(q_3+2)}{\circ} \arrow[r] & \cdots \arrow[r] & \overset{-(q_{s-1}+2)}{\circ} \arrow[r] & {[q_s-1]} \arrow[r] & \overset{-1}{\circ} \arrow[d] \\
G_s & & & & & & & \\
{[q_1-1]} \arrow[r] & \underset{-(q_s+2)}{\circ} \arrow[r] & \cdots \arrow[r] & {[q_{s-3}-1]} \arrow[r] & \underset{-(q_{s-2}+2)}{\circ} \arrow[r] & {[q_{s-1}-1]} \arrow[r] & \underset{-(q_s+1)}{\circ} \arrow[ru, phantom] & \underset{-(q_s+1)}{\circ}
\end{tikzcd}$$

if $s \equiv 0 \pmod{2}$

$$G_s \qquad \begin{array}{r} [q_0-1] \text{——} \overset{-(q_1+2)}{\circ} \text{——} [q_2-1] \text{——} \overset{-(q_3+2)}{\circ} \text{——} \cdots \text{——} \overset{-(q_{s-2}+2)}{\circ} \text{——} [q_{s-1}-1] \text{——} \overset{-(q_s+1)}{\circ} \\ \Big| \\ [q_1-1] \text{——} \underset{-(q_s+2)}{\circ} \text{——} \cdots \text{——} [q_{s-2}-1] \text{——} \underset{-(q_{s-2}+2)}{\circ} \text{——} [q_s-1] \text{——} \underset{-1}{\circ} \end{array}$$

$$\text{if } s \equiv 1 \pmod 2,$$

where $[q_i - 1]$ means a linear chain of (-2) curves of length $q_i - 1$. The (-1) curve in the middle of the graph is a cross section of the proper transform of the pencil Λ. The closure in $\mathbf{P}^2$ of a curve $y^m - \lambda x^n$ of the pencil Λ is defined by $Y^m Z^{n-m} = X^n$, where X, Y, Z are homogeneous coordinates such that $x = X/Z$ and $y = Y/Z$. We denote by the same symbol Λ the linear pencil formed by the closures in $\mathbf{P}^2$ of the members of Λ. The pencil Λ on $\mathbf{P}^2$ has base pionts at the point $(X, Y, Z) = (0, 1, 0)$ and its infinitely near points. The G-action on $\mathbf{A}^2$ extends to a G-action on $\mathbf{P}^2$, which is given by ${}^g(X, Y, Z) = (\zeta X, \zeta^e Y, Z)$, and the blowing-ups of the base points centered at $(0, 1, 0)$ are G-equivariant blowing-ups. If we eliminate these base points as well we have a nonsingular projective surface V and a $\mathbf{P}^1$-fibration $p\colon V \to C$ onto a complete curve C isomorphic to $\mathbf{P}^1$. The birational morphism $\sigma\colon V \to \mathbf{P}^2$ which eliminates the base points of the linear pencil Λ is a composite of the G-equivariant blowing-ups in the sense that the centers of the blowing-ups are the G-fixed points. We have the following assertions, which are readily verified.

LEMMA 2.4. (1) *The G-action on $\mathbf{P}^2$ extends to a G-action on V and the morphism $p\colon V \to C$ is a G-morphism with a trivial G-action on C. Hence the fibers of the $\mathbf{P}^1$-fibration p are preserved under the G-action.*

(2) *The proper transform E_∞ of the curve $x = 0$ (or equivalently $X = 0$ on $\mathbf{P}^2$) on V is a (-1) curve and it is an irreducible component with multiplicity n of a degenerate fiber F_∞ of p whose weighted graph is a linear chain consisting of the curve E_∞, the left-hand side (the upper side) Γ_∞ of the (-1) curve in the above weighted dual graph and a similar graph Δ_∞ determined uniquely by Γ_∞,*

$${}^t\Gamma_\infty \text{——} E_\infty \text{——} \Delta_\infty,$$

where ${}^t\Gamma_\infty$ is the reversed graph of Γ_∞, i.e., the vertex of Γ_∞ connected to (-1) curve in the above dual graph G_s is a tip (= an end vertex) of the dual graph of F_∞. Each component of the fibers F_0 is G-stable.

(3) *The proper transform E_0 of the curve $y = 0$ (or equivalently $Y = 0$ on $\mathbf{P}^2$) on V is a (-1) curve and it is an irreducible component with*

multiplicity m of a degenerate fiber F_0 of p whose weighted dual graph is a linear chain consisting of the curve E_0, the right-hand side (the lower side) ${}^t\Gamma_0$ of the (-1) curve in the above weighted dual graph and a similar graph Δ_0 determined uniquely by Γ_0:

$$ {}^t\Gamma_0 \text{ —— } E_0 \text{ —— } \Delta_0 $$

Each component of the fibers F_∞ is G-stable.

(4) *The (-1) curves M_0 and M_∞ arising the elimination of the base points accumulated at the points $(X, Y, Z) = (0,0,1)$, $(0,1,0)$, respectively, are the cross sections of the $\mathbf{P}^1$-fibration p. Furthermore, M_0 and M_∞ are pointwise G-fixed curves.*

(5) *Let $\overline{V}$ be the quotient V/G and let $\pi\colon V \to \overline{V}$ be the quotient morphism. Then $\overline{V}$ is a normal projective surface with a $\mathbf{P}^1$-fibration $\bar{p}$: $\overline{V} \to C$, where $\bar{p} \cdot \pi = p$, and the morphism π is a finite morphism. Furthermore, π is totally ramified over $N_0 := \pi(M_0)$ and $N_\infty := \pi(M_\infty)$ and hence $\overline{V}$ is smooth in the neighborhoods of the curves N_0 and N_∞, which are the cross sections of the $\mathbf{P}^1$-fibration $\bar{p}$. The morphism π is unramified along the fibers of p possibly except for the fibers F_0 and F_∞. Let $\overline{F_0}$ and $\overline{F_\infty}$ be the images of F_0 and F_∞ under π. Then there appear finitely many cyclic quotient singular points lying on the curves $\overline{F_0}$ and $\overline{F_\infty}$.*

3. *Construction of Étale Endomorphisms*

Now we replace the (-1) curve in the graph G_s by a $(-\ell - 1)$ curve for some positive integer ℓ and denote the modified graph by $\widetilde{G_s}(\ell)$. Then it is the resolution graph of a cyclic quotient singular point. More precisely, we have the following result. For the resolution of cyclic quotient singularity, see, for example, [3, p. 84].

Lemma 3.1. *With the above notations, let $N = mn\ell + 1$ and $e = n(m - m_1)\ell + q_0 + 1$ for some positive integer ℓ. Let a cyclic group $G = \{g^i; 0 \le i < N\}$ of order n act on the affine plane $\mathbf{A}^2 = \operatorname{Spec} k[x, y]$ by ${}^g x = \zeta x$ and ${}^g y = \zeta^e y$, where ζ is a primitive Nth root of unity. Then the resolution graph of the quotient singular point P of $\mathbf{A}^2/G$ is the graph $\widetilde{G_s}(\ell)$ given as above.*

Proof. Write the graph $\widetilde{G_s}(\ell)$ as

$$ \underset{-a_r}{\circ} \text{——} \underset{-a_{r-1}}{\circ} \text{——} \circ \text{——} \cdots \text{——} \underset{-a_2}{\circ} \text{——} \underset{-a_1}{\circ} $$

where the vertex with weight $-a_r$ is the left end vertex of $\widetilde{G_s}(\ell)$. The assertion is then equivalent to showing that the continued fraction associated with the graph $\widetilde{G_s}(\ell)$ is equal to a fraction N/e:

$$\frac{d}{e} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{r-1} - \cfrac{1}{a_r}}}}.$$

Namely, we have to show that if we write the continued fraction in the form of a single fraction N/e then the numerator N and the denominator e are given, respectively, by N and e specified in the statement. If we show the assertion that the numerator is equal to $mn\ell + 1$ then the denominator is obtained uniquely by solving the congruence equation

$$em \equiv n \pmod{mn\ell + 1} \qquad \text{and} \qquad 0 < e < mn\ell + 1.$$

Indeed, $e = n(m - m_1)\ell + q_0 + 1$ is a unique solution.

We prove the last assertion by induction on s. If $s = 1$ or 2 the assertion is straightforward. Suppose $s \geq 2$. Let $\tilde{H}$ be the graph obtained by removing the following string from $\widetilde{G_s}(\ell)$:

$$[q_0 - 1] \overset{}{\text{———}} \overset{-(q_1+2)}{\circ} \text{———} .$$

Then the reversed graph ${}^t\tilde{H}$ is the dual graph obtained from a pair (m_1, m_0) instead of (m, n). Hence, by the induction hypothesis, the continued fraction associated with ${}^t\tilde{H}$ is equal to a fraction

$$\frac{m_0 m_1 \ell + 1}{m_0(m_1 - m_2)\ell + q_1 + 1}.$$

Then the continued fraction associated with ${}^t\widetilde{G_s}(\ell)$ is obtained by making the calculations

$$\frac{a}{b} = q_1 + 2 - \frac{m_0(m_1 - m_2)\ell + q_1 + 1}{m_0 m_1 \ell + 1},$$

$$\frac{N'}{e'} = \frac{q_0 a - (q_0 - 1)b}{(q_0 - 1)a - (q_0 - 2)b},$$

where a, b, c, d are positive integers with $\gcd(a, b) = \gcd(N', e') = 1$. A straightforward computation shows that $N' = mn\ell + 1$. Then we will be done if we note the well-known fact that the continued fraction associated with $\widetilde{G_s}(\ell)$ and ${}^t\widetilde{G_s}(\ell)$ have the same numerator if they are written as single fractions with coprime numerators and denominators. Q.E.D.

Let G and its action on $\mathbf{A}^2$ be the same as in Lemma 3.1. Note that $f = y^m - x^n$ is then a semi-invariant of weight n. Letting $\rho\colon X \to B$ be the $\mathbf{A}^1{}_*$-fibration associated with the linear pencil $\{y^m = \lambda x^n\}_{\lambda \in \mathbf{P}^1}$, where $B \cong \mathbf{A}^1$. Since $B = \operatorname{Spec} k[t]$ with

$$t = \frac{y^m}{y^m - x^n}$$

and t is an invariant under the G-action, we know that the group G acts on X in such a way that G preserves each fiber of ρ. We shall show that X/G is isomorphic to X and that the quotient morphism $q\colon X \to X/G$ thereby gives rise to an étale finite endomorphism $\alpha\colon X \to X$ with degree $mn\ell + 1$.

The above G-action on $\mathbf{A}^2$ extends to a G-action on $\mathbf{P}^2$ by setting

$$(X, Y, Z) \mapsto (\zeta X, \zeta^e Y, Z).$$

The birational morphism $\sigma\colon V \to \mathbf{P}^2$ which eliminates the base points of the linear pencil Λ is, in fact, a composite of the G-equivariant blowing-ups in the sense that the centers of the blowing-ups are the G-fixed points. Furthermore, as shown in Lemma 2.4, the cross-sections M_0, M_∞ are pointwise G-fixed curves and the fibers of the $\mathbf{P}^1$-fibration $p\colon V \to C$ are preserved under the G-action. Each component of the fibers F_0 and F_∞ are G-stable. Let $\overline{V}$ be the quotient V/G and let $\pi\colon V \to \overline{V}$ be the quotient morphism. Then $\overline{V}$ is a normal projective surface with a $\mathbf{P}^1$-fibration $\bar{p}\colon \overline{V} \to C$, where $\bar{p} \cdot \pi = p$, and the morphism π is a finite morphism. Furthermore, π is totally ramified over $N_0 := \pi(M_0)$ and $N_\infty \pi(M_\infty)$ and hence $\overline{V}$ is smooth in the neighborhoods of the curves N_0 and N_∞ which are the cross-sections of the $\mathbf{P}^1$-vibration $\bar{p}$. The morphism π is unramified along the fibers of p possibly except for the fibers F_0 and F_∞. Let $\overline{F_0}$ and $\overline{F_\infty}$ be the images of F_0 and F_∞ under π. Then there appear finitely many cyclic quotient singular points lying on the curves $\overline{F_0}$ and $\overline{F_\infty}$. Let $\tau\colon W \to \overline{V}$ be the minimal resolution of these singular points and let $q = \bar{p} \cdot \tau\colon W \to C$ be the induced $\mathbf{P}^1$-fibration.

Let A_0 and A_∞ denote, respectively, the set-theoretic inverse images by τ of the union of the images by π of the irreducible components of the fibers F_0 and F_∞ corresponding to the graphs Γ_0 and Γ_∞. Then A_0 and A_∞ are the linear chains of the curves isomorphic to $\mathbf{P}^1$ because they are the parts of the degenerate fibers of the $\mathbf{P}^1$-fibration q and the corresponding parts on $\overline{V}$ are the linear chains. Let $\widetilde{\Gamma_0}$ and $\widetilde{\Gamma_\infty}$ denote, respectively, the weighted dual graphs corresponding to A_0 and A_∞. By the abuse of notations, we may write

$$\widetilde{\Gamma}_0 = \tau^{-1}(\pi(\Gamma_0)), \qquad \widetilde{\Gamma}_\infty = \tau^{-1}(\pi(\Gamma_\infty)).$$

Then Lemma 3.1 asserts that the graph

$$\widetilde{\Gamma}_\infty \text{ —— } \widetilde{N_0} \text{ —— } \widetilde{\Gamma}_0$$

with all (-1) curves and subsequently contractible curves contracted down is the resolution graph $\widetilde{G_s}(\ell)$ of the quotient singular point P of $\mathbf{A}^2/G$ under the G-action on $\mathbf{A}^2$ specified therein, where $\widetilde{N_0}$ is the proper transform of N_0 by τ. In particular, if we denote by $\overline{N_0}$ the image of $\widetilde{N_0}$ by this contraction, we have

$$\left(\overline{N_0}^2\right) = -\ell - 1, \qquad \text{while } \left(\widetilde{N_0}^2\right) = \left(N_0^2\right) = -N.$$

Let B_0 and B_∞ denote, respectively, the inverse images by τ of the union of the images by π of the irreducible components of the fibers F_0 and F_∞ corresponding to the graphs Δ_0 and Δ_∞. We contract also the (-1) curves and subsequently contractible curves contained in B_0 and B_∞. We thus obtain a birational morphism $\bar{\tau}$: $W \to \overline{W}$. Set $\overline{N_\infty} := \bar{\tau}(\widetilde{N_\infty})$, where $\widetilde{N_\infty}$ is the proper transform of N_∞ by τ, and let $\widetilde{H_s}$ be the weighted dual graph of $\bar{\tau}(B_0) + \overline{N_\infty} + \bar{\tau}(B_\infty)$.

$$\begin{tikzcd}
X \subset V \arrow[d, "\sigma"'] \arrow[dr, "\pi"] & & W \arrow[dl, "\tau"'] \arrow[dr, "\bar{\tau}"] & & W' \arrow[dl, "\tau'"] \\
X \subset \mathbf{P}^2 & \overline{V} = V/G & & \overline{W} &
\end{tikzcd}$$

The $\mathbf{P}^1$-fibration q: $W \to C$ induces a $\mathbf{P}^1$-fibration $\bar{q}$: $\overline{W} \to C$ on the surface $\overline{W}$ for which $\overline{N_0}$ and $\overline{N_\infty}$ are the cross sections of $\bar{q}$. Let F_1 be the fiber of p: $V \to C$ corresponding to the member $y^m = x^n$ of the pencil Λ on $\mathbf{P}^2$ and let $\overline{F_1}$ be the proper transform of $\pi(F_1)$ by $\bar{\tau} \cdot \tau^{-1}$. Then $\overline{F_1}$ is a smooth fiber of $\bar{q}$: $\overline{W} \to C$. Now apply the elementary transformations with centers at the point $\overline{N_\infty} \cap \overline{F_1}$ and the infinitely near points lying on $\overline{N_\infty}$. Namely, blow up this point and contract the proper transform of $\overline{F_1}$. We repeat this transformation ℓ times all together. Let τ': $W' \to W$ be a composite of the elementary transformations and let F_1' be the fiber on W' corresponding to F_1 on $\overline{W}$. Let N_0' be the proper transform of $\overline{N_0}$. Then $(N_0'^2) = -1$ and the graph $\widetilde{G_s}(\ell)$ changes back to the graph G_s. Now remove all the irreducible curves corresponding to the graphs $\widetilde{G_s}, \widetilde{H_s}$ and the fiber F_1' from W' to obtain an open set U. By the construction of W', the open set U is isomorphic to the surface X. More precisely, X is contained in the surfaces $\overline{V}$, W, $\overline{W}$, and W' and X is intact under the birational mappings τ, $\bar{\tau}$, and τ'. Hence the restriction of the quotient morphism $\pi: V \to \overline{V}$ induces an étale finite endomorphism which is a Galois covering with group G.

Combining the observations in Section 2, we have thus proved the following result.

THEOREM 3.2. *Let* $X = \operatorname{Spec} k[x, y, f^{-1}]$ *with* $f = y^m - x^n$, *where* m, n *are integers larger than* 1 *with* $\gcd(m, n) = 1$. *Let* ℓ *be a positive integer. Then every étale endomorphism* $\alpha: X \to X$ *is a finite Galois covering with Galois group* G, *which is a cyclic group of order* $mn\ell + 1$. *Furthermore, every such étale endomorphism* $\alpha: X \to X$ *is obtained as the restriction on* X *of the quotient morphism* $\mathbf{A}^2 \to \mathbf{A}^2/G$, *where the* G*-action on* $\mathbf{A}^2$ *is specified in Lemma* 3.1.

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